

JOINING MEASURES FOR HOROCYCLE FLOWS ON ABELIAN COVERS

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ABSTRACT. Let S_0 be a \mathbb{Z}^d -cover of a compact hyperbolic surface for $d = 1, 2$. We classify all locally finite joining measures for horocycle flows on the unit tangent bundle of $S_0 \times S$ for any hyperbolic surface S . We will also discuss several applications of this result.

1. INTRODUCTION

The starting point of our discussion is Ratner's joining theorem for horocycle flows on a finite volume quotient of $\mathrm{PSL}_2(\mathbb{R})$ ([29]), which is a particular case of her general classification theorem of invariant measures for unipotent flows on any finite volume homogeneous space of a connected Lie group ([30]). For infinite volume homogeneous spaces, such classification theorems are known only for some special cases (see [9], [32], [38], [3], [34], etc.)

Recently, Mohammadi-Oh ([24]) extended Ratner's joining theorem to $\Gamma \backslash G$ where $G = \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$ and Γ is a geometrically finite and Zariski dense discrete subgroup of G . In this paper, we extend Ratner's joining theorem for the unit tangent bundle of a \mathbb{Z}^d -cover of a compact hyperbolic surface.

To state our results more precisely, let $G = \mathrm{PSL}_2(\mathbb{R})$ and Γ_1, Γ_2 be discrete subgroups of G . In the whole paper, all discrete subgroups of G are assumed to be torsion-free and non-elementary. Assume further that Γ_1 is a normal subgroup of a cocompact lattice Γ'_1 of G so that $\Gamma_1 \backslash \Gamma'_1 \cong \mathbb{Z}^d$ for some positive integer d . Then $\Gamma_1 \backslash G$ is a \mathbb{Z}^d -cover of the unit tangent bundle of the compact hyperbolic surface $\Gamma'_1 \backslash \mathbb{H}^2$. For simplicity, discrete subgroups like Γ_1 will be called \mathbb{Z}^d -covers. Let

$$Z = \Gamma_1 \backslash G \times \Gamma_2 \backslash G.$$

Set

$$(1.1) \quad U = \left\{ u_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

and $\Delta(U) = \{(u_t, u_t) : t \in \mathbb{R}\}$. As is well known, the right translation action of u_t on $\Gamma_i \backslash G$ corresponds to the contracting horocycle flow when we identify $\Gamma_i \backslash G$ with the unit tangent bundle of the hyperbolic surface $\Gamma_i \backslash \mathbb{H}^2$.

Definition 1.2. Let μ_i be a locally finite U -invariant Borel measure on $\Gamma_i \backslash G$ for $i = 1, 2$. A locally finite $\Delta(U)$ -invariant measure μ on Z is called a U -joining with respect to the pair (μ_1, μ_2) if the push-forward $(\pi_i)_* \mu$ is proportional to μ_i for each $i = 1, 2$; here π_i denotes the canonical projection of Z to $\Gamma_i \backslash G$. If μ is $\Delta(U)$ -ergodic, then μ is called an ergodic U -joining.

In this paper, we investigate the U -joinings with respect to the pair of Haar measures $(m_{\Gamma_1}^{\text{Haar}}, m_{\Gamma_2}^{\text{Haar}})$. In fact, Ledrappier and Sarig showed in [20] that the Haar measure is the unique U -ergodic measure for \mathbb{Z}^d -covers which admits a generalized law of large numbers.

Our definition of U -joinings rules out the product measure $m_{\Gamma_1}^{\text{Haar}} \times m_{\Gamma_2}^{\text{Haar}}$ since its projection to $\Gamma_2 \backslash G$ is an infinite multiple of $m_{\Gamma_2}^{\text{Haar}}$. Nevertheless, a finite cover self-joining provides an example of U -joining. Recall that two subgroups of G are said to be commensurable with each other if their intersection has finite index in each of them.

Definition 1.3 (Finite cover self-joining). Suppose that for some $g_0 \in G$, Γ_1 and $g_0^{-1}\Gamma_2g_0$ are commensurable with each other. Using the map

$$\Gamma_1 \cap g_0^{-1}\Gamma_2g_0 \backslash G \rightarrow Z$$

defined by $[g] \mapsto ([g], [g_0g])$, the pushforward of the Haar measure $m_{\Gamma_1 \cap g_0^{-1}\Gamma_2g_0}^{\text{Haar}}$ to Z gives a U -joining, which will be called a finite cover self-joining. If μ is a U -joining, then any translation of μ by (e, u_t) is also a U -joining. Such a translation of a finite cover self-joining will also be called a finite cover self-joining.

Our main result is as follows:

Theorem 1.4. *Let Γ_1 be a \mathbb{Z} or \mathbb{Z}^2 -cover and let Γ_2 be any discrete subgroup of G . Then any locally finite ergodic U -joining on Z is a finite cover self-joining.*

The reason we assume Γ_1 is a \mathbb{Z} or \mathbb{Z}^2 -cover is that only for \mathbb{Z} and \mathbb{Z}^2 -covers, the geodesic flow is ergodic with respect to the Haar measures ([31]) and this property is essentially used in the proof of the main theorem.

Corollary 1.5. *Let Γ_1 be as in Theorem 1.4. Suppose Γ_2 is a discrete subgroup of G such that the U -action is ergodic on $(\Gamma_2 \backslash G, m_{\Gamma_2}^{\text{Haar}})$. Then Z admits a U -joining if and only if Γ_1 and Γ_2 are commensurable with each other, up to a conjugation.*

Under our assumption, any U -joining measure on Z can be disintegrated into an integral over a probability space of a family of U -ergodic joinings. Thus Corollary 1.5 is an immediate application of Theorem 1.4.

As an application of the joining classification theorem, we deduce the following classification of U -equivariant factor maps:

Corollary 1.6. *Let Γ be a \mathbb{Z} or \mathbb{Z}^2 -cover. Let (Y, ν) be a measure space with a locally finite U -invariant measure ν . Suppose $p : (\Gamma \backslash G, m_\Gamma) \rightarrow (Y, \nu)$*

is a U -equivariant factor map, that is, $p_*m_\Gamma = \nu$. Then (Y, ν) is isomorphic to $(\Gamma_0 \backslash G, m_{\Gamma_0})$ where Γ_0 is a discrete subgroup of G containing Γ as a finite index subgroup. Moreover, the map p can be conjugated to the canonical projection $\Gamma \backslash G \rightarrow \Gamma_0 \backslash G$.

Let A be the diagonal group in G . As another application of the joining classification theorem, we obtain a classification of $\Delta(AU)$ -invariant measures similar to [25]:

Corollary 1.7. *Let Γ_1 be a \mathbb{Z} or \mathbb{Z}^2 -cover and let Γ_2 be a cocompact lattice of G . Any $\Delta(AU)$ -invariant, ergodic, conservative, infinite Radon measure μ on $\Gamma_1 \backslash G \times \Gamma_2 \backslash G$ is one of the following:*

- (1) μ is the product measure $m_{\Gamma_1}^{\text{Haar}} \times m_{\Gamma_2}^{\text{Haar}}$;
- (2) μ is the pushforward of the Haar measure on $\Gamma_1 \cap g_0^{-1} \Gamma_2 g_0 \backslash G$ through the map:

$$\begin{aligned} \phi : \Gamma_1 \cap g_0^{-1} \Gamma_2 g_0 \backslash G &\rightarrow \Gamma_1 \backslash G \times \Gamma_2 \backslash G \\ [g] &\mapsto ([g], [g_0 g]), \end{aligned}$$

where g_0 is some element of G so that $[\Gamma_1 : \Gamma_1 \cap g_0^{-1} \Gamma_2 g_0] < \infty$.

On the proof of Theorem 1.4. Our proof is loosely modeled on Mohammadi-Oh's proof of classification of infinite U -joining measures for geometrically finite discrete subgroups ([24]). In their proof, they utilize a close relation between the Burger-Roblin measures and the Bowen-Margulis-Sullivan measures (which will be called BR measures and BMS measures respectively for short) and the finiteness of the BMS measures is crucially used. However, in our setting, both the BR measures and the BMS measures are the Haar measures and hence such a passage to finite measures is not available. Here we briefly discuss some of the main steps and difficulties.

The following window property for the horocycle flows on \mathbb{Z}^d -covers is the main technical ingredient in our approach:

Theorem 1.8. *Suppose Γ is a \mathbb{Z}^d -cover for some positive integer d . For any small $0 < \eta < 1$, there exists $0 < r = r(\eta) < 1$ such that for any non-negative $\psi \in C_c(\Gamma \backslash G)$ and for almost every $x \in \Gamma \backslash G$, there exists $T_0 = T_0(\psi, x) > 0$ so that*

$$\int_0^{rT} \psi(xu_t) dt \leq \eta \int_0^T \psi(xu_t) dt \text{ for all } T \geq T_0.$$

For a geometrically finite discrete subgroup, a similar theorem was proved in [24] using the mixing of the geodesic flow. However, this is unknown for \mathbb{Z}^d -covers. Our proof of Theorem 1.8 is built on some ideas in Ledrappier and Sarig's proof about the rational ergodicity of the horocycle flows for \mathbb{Z}^d -covers ([20], see also [35]). As an application of Theorem 1.8, we classify the orbit closures of \mathbb{Z} or \mathbb{Z}^2 -cover group in the unit tangent bundle of compact hyperbolic surfaces in the appendix (Theorem 7.9).

Now the window property enables us to show that for two nearby points x and y in Z , the set of $t \in [0, T]$ such that both $x\Delta(u_t)$ and $y\Delta(u_t)$ return to some compact subset is dynamically non-trivial as used in Hopf's ratio theorem. Based on this, we draw the following two corollaries about an arbitrary ergodic U -joining μ on Z :

- (1) almost all fibers of projection of μ on $\Gamma_1 \backslash G$ are finite;
- (2) μ is invariant under the diagonal embedding of A (up to conjugation).

Let

$$(1.9) \quad U^+ := \left\{ u_t^+ := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

be the expanding horocyclic group. Extending the invariance of μ under $\Delta(U^+)$ involves showing that a measurable $N_G(U)$ -equivariant set-valued map $\mathcal{Y} : \Gamma_1 \backslash G \rightarrow \Gamma_2 \backslash G$ is also U^+ -equivariant. The rough idea is to construct two nearby points x and $y = xu_t^+$ so that the U -orbits of $\mathcal{Y}(x)u_t^+$ and $\mathcal{Y}(y)$ do not diverge on average. Such an argument appeared in Ratner's proof of U -joining theorem for lattices in [29] (see also [12]). In [24], the finiteness of BMS measures plays an important role here: they use a Birkhoff-like equidistribution theorem for U -orbits with respect to the leaf-wise measures for the BMS measures. Differently, for \mathbb{Z}^d -covers, we deal with an infinite measure and we make the most of the Hopf's ratio theorem for horocycle flows and geodesic flows with respect to a series of compact subsets chosen with calibration.

Notational convention.

- (1) For any positive number a, b and ϵ , we write $a = e^{\pm\epsilon}b$ to mean that $e^{-\epsilon}b \leq a \leq e^\epsilon b$.
- (2) For any discrete subgroup Γ in G , denote the Haar measure on $\Gamma \backslash G$ by m_Γ . When there is no ambiguity about Γ , we simply denote it by m .

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2. SYMBOLIC DYNAMICS

For the rest of the paper, fix Γ_0 a cocompact lattice of $G = \mathrm{PSL}_2(\mathbb{R})$ and Γ a normal subgroup of Γ_0 with $\Gamma \backslash \Gamma_0 \cong \mathbb{Z}^d$ for some positive integer d . Set

$$A = \left\{ a_s := \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix} : s \in \mathbb{R} \right\}.$$

The right translation action of a_s on $\Gamma \backslash G$ corresponds to the geodesic flow on the unit tangent bundle of $\Gamma \backslash \mathbb{H}$ which can be identified with $\Gamma \backslash G$. Recall the groups U and U^+ defined in (1.1) and (1.9) respectively.

In this section, we describe the geodesic flow on $\Gamma \backslash G$ as a suspension flow, whose base is a skew product over a subshift of finite type. First recall some basic notions of symbolic dynamics.

A subshift of finite type with set of states S and transition matrix $A = (t_{ij})_{S \times S}$ ($t_{ij} \in \{0, 1\}$) is the set

$$\Sigma := \{x = (x_i) \in S^{\mathbb{Z}} : t_{x_i x_j} = 1\}$$

together with the action of the left shift map $\sigma : \Sigma \rightarrow \Sigma$, $\sigma(x)_k = x_{k+1}$ and the metric $d(x, y) = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} (1 - \delta_{x_k y_k})$. There is a one-sided version $\sigma : \Sigma^+ \rightarrow \Sigma^+$ obtained by replacing \mathbb{Z} by $\mathbb{N} \cup \{0\}$.

Suppose F is a real-valued function on Σ or Σ^+ . The Birkhoff sums of F are denoted by F_n ,

$$F_n := F + F \circ \sigma + \cdots + F \circ \sigma^{n-1}.$$

Symbolic dynamics for the geodesic flow. Fix Ω_0 to be a connected relatively compact fundamental domain in $\Gamma \backslash G$ for the left action of $\Gamma \backslash \Gamma_0$. As $\Gamma \backslash \Gamma_0 \cong \mathbb{Z}^d$, the group \mathbb{Z}^d acts on $\Gamma \backslash G$. For every $\xi \in \mathbb{Z}^d$, we denote the left action of ξ on $\Gamma \backslash G$ by D_ξ .

Definition 2.1. For every $g \in \Gamma \backslash G$, we call the unique integer $\xi(g) \in \mathbb{Z}^d$ satisfying $g \in D_{\xi(g)} \Omega_0$ the \mathbb{Z}^d -coordinate of g .

By a lifting argument of Bowen-Series symbolic dynamics of the geodesic flow on $\Gamma_0 \backslash G$ (see [8], [36], [37], [26]), we obtain the following characterization of the geodesic flow on $\Gamma \backslash G$:

Lemma 2.2. *There exist a topologically mixing two-sided subshift of finite type (Σ, σ) , a Hölder continuous function $\tau : \Sigma \rightarrow \mathbb{R}$ which depends only on the non-negative coordinates, a function $f : \Sigma \rightarrow \mathbb{Z}^d$ such that $f(x) = f(x_0, x_1)$, a Hölder function $h : \Sigma \rightarrow \mathbb{R}$ and a Hölder continuous map $\pi : \Sigma \times \mathbb{Z}^d \times \mathbb{R} \rightarrow \Gamma \backslash G$ satisfying the following properties:*

- (1) $\tau^* := \tau + h - h \circ \sigma$ is non-negative, and there exists a constant n_0 such that $\inf_{x \in \Sigma} \tau_{n_0}^*(x) > 0$.
- (2) Let

$$(\Sigma \times \{0\})_{\tau^*} := \{(x, 0, t) : x \in \Sigma, 0 \leq t < \tau^*(x)\}.$$

The restriction map $\pi : (\Sigma \times \{0\})_{\tau^} \rightarrow \Omega_0$ is a surjective finite-to-one map. Moreover, there exists a countable sequence $\{g_i\} \subseteq \Gamma \backslash G$, such that every $g \in \Gamma \backslash G$ outside $\cup_{i=1}^{\infty} g_i AU$ and $\cup_{i=1}^{\infty} g_i AU^+$ has exactly one preimage ([36]).*

- (3) *For any $(\xi_0, t_0) \in \mathbb{Z}^d \times \mathbb{R}$, define the map Q_{ξ_0, t_0} on $\Sigma \times \mathbb{Z}^d \times \mathbb{R}$ by $Q_{\xi_0, t_0}(x, \xi, t) = (x, \xi + \xi_0, t + t_0)$. Then $\pi \circ Q_{\xi_0, t_0}(x, \xi, t) = D_{\xi_0}(\pi(x, \xi, t))a_{t_0}$ for all $(x, \xi, t) \in \Sigma \times \mathbb{Z}^d \times \mathbb{R}$.*
- (4) $\pi \circ T_{f, -\tau^*} = \pi$, where $T_{f, -\tau^*}(x, \xi, t) = (\sigma x, \xi + f(x), t - \tau^*(x))$.

(5) Suppose $g = \pi(x, \xi, t)$, $g' = \pi(x', \xi', t')$. If there exist $p, q \geq 0$ such that

$$\begin{aligned} x_p^\infty &= (x')_q^\infty \text{ (i.e., } x_{p+i} = x'_{q+i} \text{ for any } i \in \mathbb{N}); \\ t - t' &= h(x) - h(x') + \tau_p(x) - \tau_q(x'); \\ \xi - \xi' &= f_q(x') - f_p(x), \end{aligned}$$

then $g' = gu_s$ for some $s \in \mathbb{R}$.

(6) Suppose $g = \pi(x, \xi, t)$, $0 \leq t < \tau^*(x)$. For every $s \in \mathbb{R}$, all but at most countably many points $g' \in gUa_s$ have a unique representation $g' = \pi(x', \xi', t')$ such that $0 \leq t' < \tau^*(x')$ and there exist p, q with $(x')_p^\infty = x_q^\infty$.

Symbolic coordinates. For every $g_i \in \Gamma \backslash G$, the point described in Lemma 2.2 (2), choose a representation $g_i = \pi(x_i, \xi_i, t_i)$ such that $0 \leq t_i < \tau^*(x_i)$. We call $(x, \xi, t) \in \Sigma \times \mathbb{Z}^d \times \mathbb{R}$ a symbolic coordinate for $g \in \Gamma \backslash G$, if

- (1) $g \notin \cup_{i=1}^\infty g_i AU$, $g = \pi(x, \xi, t)$, and $0 \leq t < \tau^*(x)$;
- (2) $g \in g_i Ua_s$, $g = \pi(x, \xi, t)$, $0 \leq t < \tau^*(x)$, and $x_p^\infty = (x_i)_q^\infty$ for some p, q .

Some points in $\Gamma \backslash G$ have more than one symbolic coordinates. But for every $g \in \Gamma \backslash G$, the set of points in gU with more than one symbolic coordinates is at most countable by Lemma 2.2 (2) and (6). In particular, for every g , the Birkhoff integral $\int_0^T f(gu_t)dt$ is determined by the t 's for which gu_t has a unique symbolic coordinate. We may therefore safely ignore the points with more than one symbolic coordinates.

Ruelle's transfer operator and the Haar measure. Consider the Ruelle's operator $L_{-\tau} : C(\Sigma^+) \rightarrow C(\Sigma^+)$ given by

$$L_{-\tau}(\varphi)(x) = \sum_{\sigma y = x} e^{-\tau(y)} \varphi(y).$$

By Ruelle-Perron-Frobenius theorem, there exist a probability measure ν' on Σ^+ and a Hölder continuous function $\psi : \Sigma^+ \rightarrow \mathbb{R}^+$ such that

$$(2.3) \quad L_{-\tau}\psi = \psi, \quad L_{-\tau}^*\nu' = \nu', \quad \text{and} \quad \int \psi d\nu' = 1.$$

The measure $\psi d\nu'$ is a shift invariant probability measure which can be extended to the two-sided shift Σ . Denote this extension by ν .

Put

$$\left(\Sigma \times \mathbb{Z}^d \right)_{\tau^*} := \{(x, \xi, t) : 0 \leq t \leq \tau^*(x)\}.$$

The following lemma is essentially in [6] (see also [3]).

Lemma 2.4. *The Haar measure on $\Gamma \backslash G$, subject to the normalization $m_\Gamma(\Omega_0) = 1$, is given by $\frac{1}{\int \tau^* d\nu} (\nu \times dm_{\mathbb{Z}^d} \times dt)|_{(\Sigma \times \mathbb{Z}^d)_{\tau^*}} \circ \pi^{-1}$.*

Symbolic local manifolds. Suppose $g \in \Gamma \backslash G$ has a symbolic coordinate (x, ξ, t) with $0 \leq t < \tau^*(x)$. Write $t = s + h(x)$. The symbolic local stable manifold of $g = \pi(x, \xi, s + h(x))$ is defined to be

$$W_{\text{loc}}^{\text{ss}}(g) := \pi\{(y, \xi, s + h(y)) : y_0^\infty = x_0^\infty\}.$$

It follows from Lemma 2.2 (5) that $W_{\text{loc}}^{\text{ss}}(g) \subset gU$. Lemma 2.2 also implies that if $W_{\text{loc}}^{\text{ss}}(g)$ intersects $W_{\text{loc}}^{\text{ss}}(g')$ with positive measure for another $g' \in \Gamma \backslash G$, then they are equal up to a set of measure 0.

Let the measure l_g on gU be given by the length measure

$$l_g(\{gu_t : a < t < b\}) = b - a.$$

Lemma 2.5 (Proposition 4.5 in [3]). *Suppose $g \in \Gamma \backslash G$ has a symbolic coordinate $(x, \xi, s + h(x))$. Then*

$$l_g[W_{\text{loc}}^{\text{ss}}(g)] = e^{-s}\psi(x_0, x_1, \dots)$$

where $\psi : \Sigma^+ \rightarrow \mathbb{R}_{>0}$ is the eigenfunction of the Ruelle's transfer operator given as (2.3).

3. WINDOW PROPERTY

Recall that Γ is a normal subgroup of a cocompact lattice Γ_0 with $\Gamma \backslash \Gamma_0 \cong \mathbb{Z}^d$ for some positive integer d .

Keep the notations in Section 2. For $g \in \Gamma \backslash G$ and $T \in \mathbb{R}$, define

$$\xi_T(g) := \xi(ga_T),$$

where $\xi(ga_T)$ is the \mathbb{Z}^d -coordinate of ga_T given as Definition 2.1.

It follows from the work of Ratner [27] and Katsuda-Sunada [16] that the distribution $\frac{\xi_T(g)}{\sqrt{T}}$ as g ranges over Ω_0 converges to the distribution of a multivariate Gaussian random variable N on \mathbb{R}^d , with a positive definite covariance matrix $\text{Cov}(N)$. Denote

$$(3.1) \quad \sigma := \sqrt[d]{|\det \text{Cov}(N)|}.$$

Consider the set

$$(3.2) \quad W := \left\{ g \in \Gamma \backslash G : \lim_{T \rightarrow \infty} \frac{\xi_T(g)}{T} = 0, \limsup_{T \rightarrow \infty} \left| \frac{\xi_T(g)}{\sqrt{T \ln \ln T}} \right| = \sqrt{2}\sigma \right\}.$$

Then W is a conull set by Corollary 6.1 in [3] and Corollary 2 in [10].

In this section, we aim to prove the window property for the horocycle flow on $\Gamma \backslash G$:

Theorem 3.3 (Window property I). *For any $0 < \eta < 1$, there exists $0 < r = r(\eta) < 1$ so that the following holds: for any $g \in W$, and for any non-negative $\psi \in C_c(\Gamma \backslash G)$, there exists $T_0 = T_0(\psi, g) > 1$ such that for every $T > T_0$ we have*

$$(3.4) \quad \int_0^{rT} \psi(gu_t) dt \leq \eta \int_0^T \psi(gu_t) dt.$$

The following is another version of window property we need in the proof of joining classification.

Theorem 3.5 (Window property II). *For any sufficiently small $0 < \delta < 1$, there exists $0 < c = c(\delta) < 1/4$ so that the following holds: for any $g \in W$ and for any non-negative $\psi \in C_c(\Gamma \backslash G)$, there exists $T_0 = T_0(\psi, g) > 1$ such that for every $T > T_0$ we have*

$$\int_T^{(1+\delta)T} \psi(gu_t) dt \leq c \int_0^T \psi(gu_t) dt.$$

3.1. Key Lemma. We show a key lemma (Lemma 3.6) leading to Theorems 3.3 and 3.5, which elaborates on the work of Ledrappier and Sarig ([20], see also [35]).

For $\varphi \in C(\Sigma^+)$, the topological pressure $P_{\text{top}}(\varphi)$ is given by

$$P_{\text{top}}(\varphi) := \sup_{\mu} \left(h_{\mu}(\sigma) + \int \varphi d\mu \right)$$

where the supremum is taken over all σ -invariant Borel probability measures μ on Σ^+ ; here $h_{\mu}(\sigma)$ denotes the measure theoretic entropy of σ with respect to μ . Let τ and f be as in Lemma 2.2. Define $P : \mathbb{R}^d \rightarrow \mathbb{R}$ implicitly by $u \mapsto P(u)$, where $P(u)$ is the root satisfying $P_{\text{top}}(-P(u)\tau + \langle u, f \rangle) = 0$. It is shown in [2] and [3] that P is a convex analytic function with $P(0) = 1$, $\nabla P(0) = 0$ and $P''(0) = \text{Cov}(N)$.

Set

$$H : \mathbb{R}^d \rightarrow \mathbb{R}$$

to be minus the Legendre transform of P . Then H is a concave analytic function with $H(0) = 1$, $\nabla H(0) = 0$ and $H''(0) = -\text{Cov}(N)^{-1}$.

Lemma 3.6 (Key Lemma). *For every small $0 < \epsilon < 1$, there exist a Borel set $E \subset \Gamma \backslash G$ of positive measure, some compact neighborhood $K = K(E, \epsilon)$ of 0 in \mathbb{R}^d and $T_0 = T_0(E, \epsilon) > 1$ so that for any $g \in \Gamma \backslash G$, if $T > T_0$ and $\frac{\xi_{T^*}(g)}{T^*} \in K$ with $T^* = \ln T$, then*

$$\int_0^T \chi_E(gu_t) dt = \frac{e^{\pm \epsilon} m_{\Gamma}(E)}{(2\pi \sigma T^*)^{\frac{d}{2}}} \cdot T \cdot \exp \left(T^* \left(H \left(\frac{\xi_{T^*}(g)}{T^*} \right) - 1 \right) \right),$$

where σ is given as (3.1).

Fix some small $\epsilon^* = \epsilon^*(\epsilon) > 0$, which will be determined later. Recall the symbolic coding introduced in Section 2, in particular the definition of the eigenfunction ψ of the Ruelle's operator ((2.3)). Denote by d_{max} the maximal diameter of a symbolic local stable manifold, measured in the intrinsic metric of the horocycle that contains it. The coding can be modified so that

$$\begin{aligned} \max \tau^* &< \epsilon^*, \quad \max |h| < \epsilon^*, \quad d_{\text{max}} < \epsilon^*, \quad \max \psi < \epsilon^*, \\ \text{diam}(\pi\{(x, \xi_0, s) : x_0 = a_0, 0 \leq s < \tau^*(x)\}) &< \epsilon^* \text{ for all } a_0, \xi_0. \end{aligned}$$

Moreover, the coding can be adjusted to satisfy the following property:

$$\frac{\max \psi}{\min \psi} < C_0,$$

where C_0 does not depend on ϵ^* or ϵ (see Section 4.1 in [35] for details).

Proof of Lemma 3.6. We divide the proof into four steps. The first three steps follow from [20], which we recall for readers' convenience.

Fix some cylinder set $[\underline{a}] = [\dot{a}_0, \dots, a_{n-1}]$ such that $\inf_{[\underline{a}]} \tau^* > 0$. Also fix some $\epsilon_0 \in (0, \inf_{[\underline{a}]} \tau^*)$ and $\xi_0 \in \mathbb{Z}^d$. Our set E is going to be

$$E := \pi(\{(x, \xi_0, t + h(x)) : x \in [\underline{a}], 0 \leq t < \epsilon_0\}).$$

For any $g \in \Gamma \backslash G$, denote $gU_T := \{gu_t : t \in [0, T]\}$. Viewing the integral $\int_0^T \chi_E(gu_t) dt$ as an integral on the horocyclic arc gU_T with respect to the measure l_g , we can write

$$\int_0^T \chi_E(gu_t) dt = l_g(E \cap gU_T) = l_g(E \cap ga_{T^*}U_1 a_{-T^*}).$$

Step 1. We approximate the horocyclic arc $ga_{T^*}U_1$ by symbolic local stable manifolds. More precisely, we claim that there exist $N^+, N^- \in \mathbb{N}$ and $g_i \in ga_{T^*}U_1$ for $i = 1, \dots, N^+$ so that setting $J_{T^*}(g_i, E) = l_g(E \cap W_{\text{loc}}^{\text{ss}}(g_i) a_{-T^*})$, we have

$$(3.7) \quad \sum_{i=1}^{N^-} J_{T^*}(g_i, E) \leq l_g(E \cap gU_T) \leq \sum_{i=1}^{N^+} J_{T^*}(g_i, E),$$

$$(3.8) \quad \left| \sum_{i=1}^{N^\pm} l(W_{\text{loc}}^{\text{ss}}(g_i)) - l(ga_{T^*}U_1) \right| \leq 4\epsilon^*.$$

In fact, this can be achieved by choosing g_i 's for $i = 1, \dots, N^-$ so that $W_{\text{loc}}^{\text{ss}}(g_i)$ is contained in $ga_{T^*}U_1$. Choose g_i 's for $i = N^- + 1, \dots, N^+$ so that $W_{\text{loc}}^{\text{ss}}(g_i)$ intersects $ga_{T^*}U_1$ with positive measure without being contained in it. Note that any two symbolic local stable manifolds are either equal or disjoint up to sets of measure 0. Therefore $l_g(E \cap gU_T)$ can be sandwiched between $\sum_{i=1}^{N^\pm} J_{T^*}(g_i, E)$ as (3.7).

The inequality (3.8) follows from the observation that every g_i lies in the d_{max} -neighborhood of $ga_{T^*}U_1$ and $d_{\text{max}} < \epsilon^*$.

Step 2. Suppose g and g_i have symbolic coordinates $(x, \xi, t + h(x))$ and $(x_i, \xi_i, t_i + h(x_i))$ respectively. Assume $T > e^{4\epsilon^*}$. Putting $T_i^\# = T^* - t_i$, it is shown in step 2 of Lemma 1 in [20] that

$$(3.9) \quad J_{T^*}(g_i, E) = e^{\pm \epsilon_0} \sum_{k=0}^{\infty} \sum_{\sigma^k y = (x_i)_0^\infty} \chi_{[0, \epsilon_0]}(r_k(y) - T_i^\#) \delta_{\xi_i - \xi_0}(f_k(y)) \chi_{[\underline{a}]}(y) \psi(y),$$

where the y 's in this sum take values in the one-sided shift Σ^+ .

We note for future reference that $|T_i^\# - T^*| = |t_i| < \max \tau^* + \max |h| < 2\epsilon^*$.

Step 3. Using an elaboration of Lalley's method ([18]), it is proved in the appendix of [20] that there exists a compact neighborhood \tilde{K}_0 of 0 in \mathbb{R}^d and $T_0 > 1$ depending on E and ϵ^* so that for every $T > T_0$ and every i , if $\frac{\xi_i}{T_i^\#} \in \tilde{K}_0$, then

$$(3.10) \quad J_{T^*}(g_i, E) = \frac{e^{\pm 10\epsilon^*}}{(2\pi\sigma T^*)^{d/2}} \cdot \exp\left(T_i^\# H\left(\frac{\xi_i}{T_i^\#}\right)\right) \cdot m(E) \cdot \psi(x_i),$$

where σ is defined as (3.1).

Step 4. Now (3.7), (3.9) and (3.10) together imply that

$$l_g(E \cap gU_T) \leq \frac{e^{10\epsilon^*}}{(2\pi\sigma T^*)^{d/2}} \cdot m(E) \sum_{i=1}^{N^+} \exp\left(T_i^\# H\left(\frac{\xi_i}{T_i^\#}\right)\right) \psi(x_i).$$

We compare $T^* H\left(\frac{\xi_{T^*}(g)}{T^*}\right)$ with $T_i^\# H\left(\frac{\xi_i}{T_i^\#}\right)$. Without loss of generality, assume \tilde{K}_0 is sufficiently small so that for every $x \in \tilde{K}_0$, we have $|H(x) - H(0)| < \epsilon^*$ and $\|\nabla H(x) - \nabla H(0)\| < \epsilon^*$, where $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^d . Let K_0 be some smaller compact neighborhood of 0 inside \tilde{K}_0 with $\text{diam}(K_0) < \frac{1}{4} \text{diam}(\tilde{K}_0)$.

Suppose $\frac{\xi_{T^*}(g)}{T^*} \in K_0$. By construction, all the g_i 's belong to a d_{\max} -neighborhood of $A_1(ga_{T^*})$, a horocyclic arc of length 1. Their \mathbb{Z}^d -coordinates $\xi_i = \xi(g_i)$ must therefore be within a bounded distance D from each other and that of ga_{T^*} . As a result, if T is large enough, then $\frac{\xi_{T^*}(g)}{T^*} \in K_0$ implies that $\frac{\xi_i}{T_i^\#} \in \tilde{K}_0$. Estimate the difference

$$\begin{aligned} & \left| T^* H\left(\frac{\xi_{T^*}(g)}{T^*}\right) - T_i^\# H\left(\frac{\xi_i}{T_i^\#}\right) \right| \\ & \leq |T^* - T_i^\#| \cdot \left| H\left(\frac{\xi_{T^*}(g)}{T^*}\right) \right| + |T_i^\#| \cdot \left| H\left(\frac{\xi_{T^*}(g)}{T^*}\right) - H\left(\frac{\xi_i}{T_i^\#}\right) \right| \\ & \leq 2\epsilon^* \cdot |H(0) + \epsilon^*| + (1 + \epsilon^*) \cdot T^* \cdot (\|\nabla H(0)\| + \epsilon^*) \cdot \left\| \frac{\xi_{T^*}(g)}{T^*} - \frac{\xi_i}{T_i^\#} \right\|. \end{aligned}$$

Since T is large and $\frac{\xi_i}{T_i^\#} \in \tilde{K}_0$, we have

$$\begin{aligned} T^* \cdot \left\| \frac{\xi_{T^*}(g)}{T^*} - \frac{\xi_i}{T_i^\#} \right\| & \leq T^* \cdot \left\| \frac{\xi_{T^*}(g)}{T^*} - \frac{\xi_i}{T^*} \right\| + T^* \cdot \left\| \frac{\xi_i}{T^*} - \frac{\xi_i}{T_i^\#} \right\| \\ & \leq D + 2\epsilon^* \cdot \text{diam } \tilde{K}_0. \end{aligned}$$

Viewing that $H(0) = 1$ and $\nabla H(0) = 0$, there exists a constant $k > 0$ independent of ϵ^* so that for any T large enough, if $\frac{\xi_{T^*}(g)}{T^*} \in K_0$, then

$$\exp \left| T^* H \left(\frac{\xi_{T^*}(g)}{T^*} \right) - T_i^\# H \left(\frac{\xi_i}{T_i^\#} \right) \right| \leq \exp k\epsilon^*.$$

Consequently, we get an upper bound for $l_g(E \cap gU_T)$:

$$l_g(E \cap gU_T) \leq \frac{e^{(10+k)\epsilon^*}}{(2\pi\sigma T^*)^{d/2}} \cdot m(E) \cdot \exp \left(T^* H \left(\frac{\xi_{T^*}(g)}{T^*} \right) \right) \cdot \left(\sum_{i=1}^{N^+} \psi(x_i) \right).$$

For the sum of $\psi(x_i)$'s, Lemma 2.5 yields

$$\psi(x_i) = e^{t_i} l_{g_i}(W_{\text{loc}}^{\text{ss}}(g_i)) = e^{\pm 2\epsilon^*} l_{g_i}(W_{\text{loc}}^{\text{ss}}(g_i)).$$

It follows from (3.8) that

$$\sum_{i=1}^{N^+} \psi(x_i) \leq e^{2\epsilon^*} \sum_{i=1}^{N^+} l_{g_i}(W_{\text{loc}}^{\text{ss}}(g_i)) \leq e^{6\epsilon^*}.$$

Letting $\epsilon^* = \epsilon/(16+k)$, we show the upper bound for $l_g(E \cap gU_T)$.

The lower bound can be obtained in a similar way. The proof is completed. \square

3.2. Proof of the window property I, II. Recall the following result about generic points for the horocycle flow for \mathbb{Z}^d -covers.

Definition 3.11. Suppose $\phi^t : X \rightarrow X$ is a continuous flow on a second countable and locally compact metric space X . A point $x \in X$ is called generic for a ϕ^t -invariant Radon measure μ , if for all $f, g \in C_c(X)$ with nonzero integrals,

$$\lim_{T \rightarrow \infty} \frac{\int_0^T f(\phi^t x) dt}{\int_0^T g(\phi^t x) dt} = \frac{\int f d\mu}{\int g d\mu}.$$

Theorem 3.12 (Sarig-Shapira [35]). *A point $g \in \Gamma \backslash G$ is generic for the horocycle flow with respect to the Haar measure m_Γ if and only if $\lim_{T \rightarrow \infty} \frac{\xi_T(g)}{T} = 0$. In particular, every point in W (given as (3.2)) is generic.*

Proof of Theorem 3.3. Fix $0 < \eta < 1$ and some small $0 < \epsilon < 1$ (which will be determined later). Let E be the set given by Lemma 3.6 for ϵ . We claim that there exists $0 < r = r(\eta) < 1$ such that for every $g \in W$, there exists $T_0 = T_0(g, \psi)$ so that for every $T > T_0$, we have

$$(3.13) \quad \int_0^{rT} \chi_E(gu_t) dt \leq \eta \int_0^T \chi_E(gu_t) dt.$$

In view of Lemma 3.6, it suffices to show the existence of r satisfying the inequality

$$\begin{aligned} & \frac{e^\epsilon m(E)}{(2\pi\sigma(rT)^*)^{d/2}} \cdot \exp\left((rT)^* H\left(\frac{\xi_{(rT)^*}(g)}{(rT)^*}\right)\right) \\ & \leq \eta \cdot \frac{e^{-\epsilon} m(E)}{(2\pi\sigma T^*)^{d/2}} \cdot \exp\left(T^* H\left(\frac{\xi_{T^*}(g)}{T^*}\right)\right), \end{aligned}$$

or equivalently the inequality

(3.14)

$$\exp\left((rT)^* H\left(\frac{\xi_{(rT)^*}(g)}{(rT)^*}\right) - T^* H\left(\frac{\xi_{T^*}(g)}{T^*}\right)\right) \leq \eta \cdot e^{-2\epsilon} \cdot \left(\frac{(rT)^*}{T^*}\right)^{d/2},$$

where $T^* = \ln T$.

The key to obtain such r is to estimate the upper bound for the following difference. Since $\frac{\xi_{T^*}(g)}{T^*} \rightarrow 0$, using the Taylor expansion for H , we have for any sufficiently large T

$$\begin{aligned} & (rT)^* H\left(\frac{\xi_{(rT)^*}(g)}{(rT)^*}\right) - T^* H\left(\frac{\xi_{T^*}(g)}{T^*}\right) \\ & = (rT)^* \left(H(0) + \frac{1}{2} \left(\frac{\xi_{(rT)^*}(g)}{(rT)^*} \right)^\top H''(0) \left(\frac{\xi_{(rT)^*}(g)}{(rT)^*} \right) + O\left(\left\| \frac{\xi_{(rT)^*}(g)}{(rT)^*} \right\|^3\right) \right) \\ & \quad - T^* \left(H(0) + \frac{1}{2} \left(\frac{\xi_{T^*}(g)}{T^*} \right)^\top H''(0) \left(\frac{\xi_{T^*}(g)}{T^*} \right) + O\left(\left\| \frac{\xi_{T^*}(g)}{T^*} \right\|^3\right) \right) \\ & = \ln r + \frac{1}{2} \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right)^\top H''(0) \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right) + O\left(\frac{\|\xi_{(rT)^*}(g)\|^3}{((rT)^*)^2}\right) \\ & \quad - \frac{1}{2} \left(\frac{\xi_{T^*}(g)}{\sqrt{T^*}} \right)^\top H''(0) \left(\frac{\xi_{T^*}(g)}{\sqrt{T^*}} \right) + O\left(\frac{\|\xi_{T^*}(g)\|^3}{(T^*)^2}\right). \end{aligned}$$

We analyze the above sum term by term. Noting that $H''(0) = -(\text{Cov}(N))^{-1}$ with $\text{Cov}(N)$ positive definite, we have

$$\begin{aligned} & \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right)^\top H''(0) \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right) - \left(\frac{\xi_{T^*}(g)}{\sqrt{T^*}} \right)^\top H''(0) \left(\frac{\xi_{T^*}(g)}{\sqrt{T^*}} \right) \\ & = \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} + \frac{\xi_{T^*}(g)}{\sqrt{T^*}} \right)^\top H''(0) \left(\frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} - \frac{\xi_{T^*}(g)}{\sqrt{T^*}} \right) \\ & = \left(\sqrt{T^*} \cdot \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} - \xi_{T^*}(g) \right)^\top H''(0) \left(\frac{\xi_{T^*}(g)}{T^*} + \frac{1}{\sqrt{T^*}} \cdot \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right) \\ & \leq C \cdot \left\| \xi_{T^*}(g) - \sqrt{T^*} \cdot \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right\| \cdot \left\| \frac{\xi_{T^*}(g)}{T^*} + \frac{1}{\sqrt{T^*}} \cdot \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right\|, \end{aligned}$$

where $C > 0$ is some constant only depending on $\text{Cov}(N)$.

Since ga_{T^*} is at most $-\ln r$ away from $ga_{(rT)^*}$, we have $\|\xi_{T^*}(g) - \xi_{(rT)^*}(g)\| \leq -\ln r/M + 2$, where $M := \text{diam}(\Omega_0)$. Then utilizing the property that $\limsup_{T \rightarrow \infty} \left| \frac{\xi_T(g)}{\sqrt{T \ln \ln T}} \right| = \sqrt{2}\sigma$, we have for any large T ,

$$\begin{aligned} & \left\| \xi_{T^*}(g) - \sqrt{T^*} \cdot \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right\| \\ & \leq \|\xi_{T^*}(g) - \xi_{(rT)^*}(g)\| + \left\| \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \cdot (\sqrt{T^*} - \sqrt{(rT)^*}) \right\| \\ & \leq -\frac{\ln r}{M} + 2 + \left\| \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^* \ln \ln((rT)^*)}} \right\| \cdot |(\sqrt{T^*} - \sqrt{(rT)^*}) \cdot \ln \ln((rT)^*)| \\ & \leq -\frac{\ln r}{M} + 2 + 3\sigma \cdot \left| \frac{-\ln r \cdot \ln \ln((rT)^*)}{\sqrt{T^*} + \sqrt{(rT)^*}} \right| \\ & \leq -\ln r \left(\frac{1}{M} + \epsilon \right) + 2. \end{aligned}$$

Meanwhile, applying the property that $\lim_{T \rightarrow \infty} \frac{\xi_T(g)}{T} = 0$, we obtain

$$\left\| \frac{\xi_{T^*}(g)}{T^*} + \frac{1}{\sqrt{T^*}} \cdot \frac{\xi_{(rT)^*}(g)}{\sqrt{(rT)^*}} \right\| \leq \left\| \frac{\xi_{T^*}(g)}{T^*} \right\| + \left\| \frac{\xi_{(rT)^*}(g)}{(rT)^*} \right\| \cdot \frac{\sqrt{(rT)^*}}{\sqrt{T^*}} \leq 3\epsilon.$$

For the higher degree terms, we have the estimate:

$$\frac{\|\xi_{T^*}(g)\|^3}{(T^*)^2} = \left\| \frac{\xi_{T^*}(g)}{\sqrt{T^* \ln \ln T^*}} \right\|^3 \cdot \sqrt{\frac{(\ln \ln T^*)^3}{T^*}} \rightarrow 0.$$

As a result, when ϵ is appropriately chosen, for any large $T > 0$, we obtain an upper bound:

$$(rT)^* H \left(\frac{\xi_{(rT)^*}(g)}{(rT)^*} \right) - T^* H \left(\frac{\xi_{T^*}(g)}{T^*} \right) \leq \sqrt{r} \cdot e^{(3C+2)\epsilon}.$$

Since $\left(\frac{(rT)^*}{T^*} \right)^{d/2} \rightarrow 0$ as $T \rightarrow \infty$, if $0 < r < 1$ satisfies

$$\sqrt{r} e^{(3C+2)\epsilon} \leq \frac{1}{2} \cdot \eta \cdot e^{-2\epsilon},$$

then such r satisfies (3.14).

Now recall that every point in W is generic for the horocycle flow (Theorem 3.12). For a general non-negative function $\psi \in C_c(\Gamma \backslash G)$ and for any $g \in W$, we have

$$\lim_{T \rightarrow \infty} \frac{\int_0^T \psi(gu_t) dt}{\int_0^T \chi_E(gu_t) dt} = \frac{\int \psi dm_\Gamma}{\int \chi_E dm_\Gamma}.$$

This limit together with (3.13) yield (3.4). \square

Proof of Theorem 3.5. Fix $0 < \delta < 1$ and some small $0 < \epsilon < 1$ to be determined later. Let E be the set given by Lemma 3.6 for ϵ . We just need to show Theorem 3.5 holds for χ_E and the general statement follows from

Hopf's ratio theorem. In view of Lemma 3.6, it suffices to show the existence of c satisfying the following inequality:

$$(3.15) \quad \frac{e^\epsilon m(E)}{(2\pi\sigma((1+\delta)T^*)^{d/2})} \cdot \exp\left(\left((1+\delta)T\right)^* H\left(\frac{\xi_{((1+\delta)T)^*}(g)}{((1+\delta)T)^*}\right)\right) \\ \leq (1+c) \frac{e^{-\epsilon} m(E)}{(2\pi\sigma T^*)^{d/2}} \exp\left(T^* H\left(\frac{\xi_{T^*}(g)}{T^*}\right)\right).$$

Using the same argument as the proof of Theorem 3.3, we obtain an upper bound for the following difference for T large enough:

$$\left((1+\delta)T\right)^* H\left(\frac{\xi_{((1+\delta)T)^*}(g)}{((1+\delta)T)^*}\right) - T^* H\left(\frac{\xi_{T^*}(g)}{T^*}\right) \\ \leq \frac{3}{2} \ln(1+\delta) + (3C+2)\epsilon,$$

where C is a constant just depending on $\text{Cov}(N)$.

Since $\left(\frac{((1+\delta)T)^*}{T^*}\right)^{d/2} \rightarrow 1$ as $T \rightarrow \infty$, if $0 < c = c(r) < 1/4$ satisfies

$$(1+\delta)^{3/2} e^{(3C+2)\epsilon} < (1+c) e^{-2\epsilon},$$

then such c makes (3.15) hold. \square

Remark 3.16. It can be deduced from the proof that given any non-negative $\psi \in C_c(\Gamma \backslash G)$ and any compact set $\Omega \subset \Gamma \backslash G$, Theorems 3.3 and 3.5 can be made uniform on Ω if $\frac{\xi_T(\cdot)}{T}$, $\sup_{t \geq T} \left| \frac{\xi_t(\cdot)}{t \ln \ln t} \right|$ and $\frac{\int_0^T u_t \cdot \psi(\cdot) dt}{\int_0^T u_t \cdot \chi_E(\cdot) dt}$ converge uniformly on Ω .

4. WEAK (C, α) -GOOD PROPERTY FOR \mathbb{Z}^d -COVERS

Recall that Γ is a \mathbb{Z}^d -cover for some positive integer d . The following terminology is introduced in [17].

Definition 4.1. Let $C, \alpha > 0$ and denote the Lebesgue measure on \mathbb{R} by $|\cdot|$. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be (C, α) -good on \mathbb{R} if for any interval $J \subset \mathbb{R}$ and $\epsilon > 0$ one has

$$|\{x \in J : |f(x)| < \epsilon\}| \leq C \cdot \left(\frac{\epsilon}{\sup_{x \in J} |f(x)|} \right)^\alpha \cdot |J|.$$

It follows from Lagrange's interpolation that if f is a polynomial of degree not greater than k , then f is $(k(k+1)^{1/k}, 1/k)$ -good on \mathbb{R} .

We prove a weak form of (C, α) -good property of polynomials which is related to the recurrence of the horocycle flow $\Gamma \backslash G$. For any positive integer k , denote by \mathcal{P}_k the set of polynomials $\Theta : U \rightarrow \mathbb{R}$ of degree at most k .

Lemma 4.2. Fix $k \geq 1$. For any compact set $K \subset \Gamma \backslash G$ and any small $0 < \epsilon < 1$, there exists a constant $0 < C < 1$ (independent of K and ϵ), a

compact subset $K_0 \subset K$ with $m(K_0) \geq (1 - \epsilon)m(K)$ and $T_0 = T_0(K_0) > 1$ so that the following inequality holds for every $g \in K_0$, $T > T_0$ and $\Theta \in \mathcal{P}_k$:

$$(4.3) \quad \int_0^T \chi_K(gu_t) |\Theta(t)| dt \geq C \cdot \int_0^T \chi_K(gu_t) dt \cdot \sup_{t \in [0, T]} |\Theta(t)|.$$

Proof. Fix K and ϵ . By Theorem 3.3 and Remark 3.16, there exist $0 < r_0 = r_0(1/2k) < 1$, $T_0 > 1$, a compact set $K' \subset K$ with $m(K') > (1 - \epsilon/2)m(K)$ and $T_0 = T_0(K') > 1$ such that for every $T > T_0$ and every $g \in K'$, we have

$$(4.4) \quad \int_0^{r_0 T} \chi_K(gu_t) dt \leq \frac{1}{2k} \int_0^T \chi_K(gu_t) dt.$$

Fix some sufficiently small $\delta > 0$. By Theorem 3.5 together with Remark 3.16, there exist $0 < c = c(\delta) < 1$, a compact subset $K_0 \subset K'$ with $m(K_0) > (1 - \epsilon/2)m(K')$ and $T'_0 = T'_0(K_0) > 1$ so that the following inequalities hold for every $g \in K_0$ and every $T > T'_0$:

$$(4.5) \quad \begin{aligned} \int_T^{(1+l(\delta))T} \chi_K(gu_t) dt &\leq \frac{c}{2k} \int_0^T \chi_K(gu_t) dt, \\ \int_{(1-l(\delta))T}^T \chi_K(gu_t) dt &\leq \frac{c}{2k} \int_0^T \chi_K(gu_t) dt, \end{aligned}$$

where $l(\delta) := r_0^{-1}k(k+1)^{1/k}\delta^{1/k}$.

We show every $g \in K_0$ satisfies (4.3). Fix any $T > \max\{T_0, T'_0/r_0\}$. We claim that there exists a constant $C_T \in (0, 1)$ such that for every $g \in K_0$ and every $\Theta \in \mathcal{P}_k$, we have

$$(4.6) \quad \int_0^T \chi_K(gu_t) |\Theta(t)| dt \geq C_T \cdot \int_0^T \chi_K(gu_t) dt \cdot \sup_{t \in [0, T]} |\Theta(t)|.$$

It can be seen from the process that C_T can be chosen independent of T , K and ϵ .

By multiplying both sides of (4.6) by scalar if necessary, it suffices to verify (4.6) holds for every polynomial in $\mathcal{P}_k^1 = \{\Theta \in \mathcal{P}_k : \sup_{[0, T]} |\Theta(t)| = 1\}$.

Let $\Theta \in \mathcal{P}_k^1$. The potential obstacle to obtain (4.6) is the following set

$$I_\Theta := \{t \in [0, T] : |\Theta(t)| < \delta\}.$$

The (C, α) -good property of polynomials on \mathbb{R} implies that

$$|I_\Theta| \leq k(k+1)^{1/k}\delta^{1/k}T.$$

Then

$$\int_0^T \chi_K(gu_t) |\Theta(t)| dt \geq \delta \cdot \int_{[0, T] \setminus I_\Theta} \chi_K(gu_t) dt.$$

As a result, (4.6) follows if there exists $0 < C'_T < 1$ such that

$$(4.7) \quad \int_{I_\Theta} \chi_K(gu_t) dt \leq C'_T \cdot \int_0^T \chi_K(gu_t) dt.$$

Since Θ is a polynomial of degree at most k , I_Θ consists of at most k intervals with the length of each interval less than $k(k+1)^{1/k}\delta^{1/k}T$. Let I be one of these intervals. There are two cases to discuss.

Case 1. Suppose $I \subset [0, r_0T]$. Then it follows from (4.4) that

$$\int_I \chi_K(gu_t)dt \leq \int_0^{r_0T} \chi_K(gu_t)dt \leq \frac{1}{2k} \int_0^T \chi_K(gu_t)dt.$$

Case 2. There exists $t_0 \in I \cap (r_0T, T]$. Recalling that $l(\delta) = r_0^{-1}k(k+1)^{1/k}\delta^{1/k}$, we have

$$\begin{aligned} I &\subset [t_0 - k(k+1)^{1/k}\delta^{1/k}T, t_0 + k(k+1)^{1/k}\delta^{1/k}T] \\ &\subset [(1-l(\delta))t_0, (1+l(\delta))t_0]. \end{aligned}$$

Applying (4.5), we have

$$\int_I \chi_K(gu_t)dt \leq \int_{(1-l(\delta))t_0}^{(1+l(\delta))t_0} \chi_K(gu_t)dt \leq \frac{c}{k} \int_0^{t_0} \chi_K(gu_t)dt.$$

Therefore (4.7) holds for $C'_T = k \cdot \max\{\frac{c}{k}, \frac{1}{2k}\}$. Noting that C'_T does not depend on T , K and ϵ , the proof of the lemma is completed. \square

5. RIGIDITY OF AU -EQUIVARIANT MAPS

For the rest of the paper, let Γ_1 and Γ_2 be discrete subgroups of G . Denote $\Gamma_i \backslash G$ by X_i . Assume Γ_1 is a \mathbb{Z} or \mathbb{Z}^2 -cover. Let

$$\varphi_1, \dots, \varphi_k : X_1 \rightarrow X_2$$

be Borel measurable maps such that for any two distinct i, j , we have $\varphi_i \neq \varphi_j$ almost everywhere. Define the set-valued map:

$$\Phi(x) = \{\varphi_1(x), \dots, \varphi_k(x)\}.$$

This section is devoted to showing the rigidity of AU -equivariant maps.

Theorem 5.1. *Suppose that there exists a conull set $X' \subset X_1$ such that for every $x \in X'$ and every $a_s u_t \in AU$, we have*

$$\Phi(x a_s u_t) = \Phi(x) a_s u_t.$$

Then there exists a conull set $X'' \subset X'$ such that for all $x \in X''$ and for every $u_r^+ \in U^+$ with $x u_r^+ \in X''$, we have

$$(5.2) \quad \Phi(x u_r^+) = \Phi(x) u_r^+.$$

The proof is inspired by the previous works [28], [13] and [24]. Different from their setting, we now need to deal with infinite measures and make use of Hopf's ratio theorem instead of Birkhoff ergodic theorem.

5.1. Reduction of Theorem 5.1.

Lemma 5.3. *Theorem 5.1 holds if there exists a conull set $\tilde{X} \subset X'$ and $r_0 > 0$ such that for every $x \in \tilde{X}$ and every $r \in (-r_0, r_0)$ with $xu_r^+ \in \tilde{X}$,*

$$\Phi(xu_r^+) = \Phi(x)u_r^+.$$

Proof. Set

$$X'' := \left\{ x \in \tilde{X} : \int_0^\infty \chi_{\tilde{X}^c}(xa_{-s})ds = 0 \right\}.$$

Then X'' is a conull set of \tilde{X} . We show X'' satisfies Theorem 5.1.

Fix any $x \in X''$ and $u_r^+ \in U^+$ with $xu_r^+ \in X''$. We may assume that $r > 0$. The property of X'' implies there exists $s > 0$ large enough so that $e^{-s}r < r_0$ and $xa_{-s}, xu_r^+a_{-s} \in \tilde{X}$. Then Lemma 5.3 can be deduced from a series of equivalent relations:

$$\begin{aligned} \Phi(xu_r^+) &= \Phi(x)u_r^+ \\ \iff \Phi(xu_r^+)a_{-s} &= \Phi(x)a_{-s}u_{e^{-s}r}^+ \\ \iff \Phi(xu_r^+a_{-s}) &= \Phi(xa_{-s})u_{e^{-s}r}^+ & (\text{by the } A\text{-equivariance}) \\ \iff \Phi(xa_{-s}u_{e^{-s}r}^+) &= \Phi(xa_{-s})u_{e^{-s}r}^+ & (\text{by the property of } \tilde{X}). \end{aligned}$$

□

5.2. Key proposition for Theorem 5.1. Recall the polynomial divergence of horocycle flow. It is known (see for example [12]) that there are universal constants $\rho_0 \in (0, 1)$, $C_0 > 1$ and $n_0 \in \mathbb{N}_+$ so that for all $x, y \in G$ and any interval $I \subset \mathbb{R}$ on which

$$(d(xu_t, yu_t))^2 < \rho_0^2,$$

there exists a polynomial P of degree at most n_0 such that

$$P(s)/C_0 \leq (d(xu_t, yu_t))^2 \leq C_0 P(s)$$

for all $s \in I$.

We introduce three compact sets $K \subset \Omega \subset Q$ in X_1 .

Construction of Q . Fix some small $\epsilon_1 > 0$. Choose a compact set Q in X_1 so that there exists a symmetric neighborhood U of e in G satisfying:

$$m(\cup_{u \in U} Qu \setminus \cap_{u \in U} Qu) < \epsilon_1 m(Q).$$

Denote

$$(5.4) \quad Q^+ = \cup_{u \in U} Qu \quad \text{and} \quad Q^- = \cap_{u \in U} Qu.$$

Construction of $\Omega \subset Q$. Let Ω be a compact subset of Q satisfying the following properties:

- $\Omega \subset X'$ (X' is given as Theorem 5.1).
- $m(\Omega) > (1 - \epsilon_1)m(Q)$.
- If $i \neq j$, then $\varphi_i(x) \neq \varphi_j(x)$ for every $x \in \Omega$.
- For every $i \in \{1, \dots, k\}$, we have φ_i continuous on Ω .

In view of the properties of Ω , there exists $\rho \in (0, \min\{\epsilon_1, \rho_0\})$ such that for every $x \in \Omega$, if $i \neq j$, then

$$(5.5) \quad d(\varphi_i(x), \varphi_j(x)) > 2\rho.$$

Set

$$\mathcal{F}_1 := \{\chi_\Omega, \chi_Q, \chi_{Q^+}, \chi_{Q^-}\}.$$

Construction of $K \subset \Omega$. Let K be a compact subset in Ω satisfying the following properties:

- $m(K) > (1 - \epsilon_1)m(\Omega)$.
- Lemma 4.2 holds for χ_Ω on K with constants C_1 (independent of Ω , K and ϵ_1) and T_0 .
- Hopf's ratio theorem for the horocycle flow holds uniformly on K for the family of functions in \mathcal{F}_1 .

Let $T_1 > 0$ be the starting point such that for every $T > T_1$, every $x \in K$, and every $f_1, f_2 \in \mathcal{F}_1$, we have

$$(5.6) \quad \frac{\int_0^T f_1(xu_t)dt}{\int_0^T f_2(xu_t)dt} > (1 - \epsilon_1) \frac{m(f_1)}{m(f_2)}.$$

Since C_0 and C_1 are independent of Ω , K and ϵ_1 , we may assume

$$(5.7) \quad (1 - \epsilon_1)^5 > \max\left\{\frac{3}{4}, 1 - \frac{C_1}{4C_0^2}\right\}.$$

Set

$$\mathcal{F}_2 := \{\chi_K, \chi_Q, \chi_{Q^+}, \chi_{Q^-}\}.$$

Construction of conull set $\tilde{X} \subset X'$. Let \tilde{X} be a conull subset in X' satisfying the following properties:

- for every $x \in \tilde{X}$, we have

$$\int_0^\infty \chi_K(xa_{-s})ds = \infty.$$

- Hopf's ratio theorem for the geodesic flow holds for every point in \tilde{X} for the family of functions in \mathcal{F}_2 .

We will show that there exists $r_0 > 0$ such that for every $x \in \tilde{X}$ and every $r \in (0, r_0)$ with $xu_r^+ \in \tilde{X}$,

$$\Phi(xu_r^+) = \Phi(x)u_r^+.$$

We first prove the following intermediate result:

Proposition 5.8. *Under the hypothesis of Theorem 5.1, there exists $r_0 > 0$ such that for every $x \in \tilde{X}$, $r \in (0, r_0)$ with $xu_r^+ \in \tilde{X}$, and for every $s > \max\{T_0, T_1\}$, if $xa_{-s}, xu_r^+a_{-s} \in K$, then*

$$\Phi(xu_r^+)u_{-r}^+ \subset \Phi(x) \cdot \{g \in G : d(g, e) \leq c \cdot e^{-s}\},$$

where $c > 1$ is an absolute constant.

Proof. Fix $x \in \tilde{X}$. For every $r > 0$ and $s > \max\{T_0, T_1\}$, if $xu_r^+ \in \tilde{X}$ then

$$\Phi(x)a_{-s} = \Phi(xa_{-s})$$

and

$$\Phi(xu_r^+)u_{-r}^+a_{-s} = \Phi(xu_r^+a_{-s})u_{-e^{-s}r}^+ = \Phi(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+.$$

We compare the distance between the U -orbits of $\Phi(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+$ and $\Phi(xa_{-s})$ and show that they do not diverge on average.

Step 1. Let ρ be given as (5.5). There exist $\epsilon_2 \in (0, \rho/2)$ and $\epsilon_3 \in (0, \epsilon_2)$ such that for every $r \in (0, \epsilon_3)$ and $s > \max\{T_0, T_1\}$, if $xa_{-s}, xu_r^+a_{-s}(= xa_{-s}u_{e^{-s}r}^+) \in K$, then

$$d(\varphi_i(xu_r^+a_{-s})u_{-e^{-s}r}^+, \varphi_i(xa_{-s})) < 2\epsilon_2.$$

Moreover we have for every $t \in [0, \max\{T_0, T_1\}]$

$$d(\varphi_i(xu_r^+a_{-s})u_{-e^{-s}r}^+u_t, \varphi_i(xa_{-s})u_t) < \rho,$$

where ρ is the constant given as (5.5).

Since φ_i is continuous on Ω , there exists $\epsilon_2 \in (0, \rho/2)$ such that for each i and for every $x, y \in \Omega$ if

$$d(\varphi_i(x), \varphi_i(y)) < 2\epsilon_2,$$

then for all $t \in [0, \max\{T_0, T_1\}]$,

$$d(\varphi_i(x)u_t, \varphi_i(y)u_t) < \rho,$$

where ρ is the constant given as (5.5).

Let $\epsilon_3 \in (0, \epsilon_2)$ be a constant so that for every $x, y \in \Omega$, if

$$d(x, y) < \epsilon_3,$$

then

$$d(\varphi_i(x), \varphi_i(y)) < \epsilon_2.$$

Consequently, for any $r \in (0, \epsilon_3)$ and $s > \max\{T_0, T_1\}$, if xa_{-s} and $xu_r^+a_{-s} \in K$, then

$$d(\varphi_i(xu_r^+a_{-s})u_{-e^{-s}r}^+, \varphi_i(xa_{-s})) < 2\epsilon_2,$$

and the second inequality follows from the choice of ϵ_2 .

In view of (5.7), we can let ϵ_2 small enough such that

$$(5.9) \quad 4\epsilon_2^2 + 2\rho^2(1 - (1 - \epsilon_1)^5(1 - \epsilon_2)^2) < \frac{C_1\rho^2}{2C_0^2}.$$

For the rest of the proof, we fix any $s > \max\{T_0, T_1\}$ and any $r \in (0, \epsilon_3)$ such that $xa_{-s}, xu_r^+a_{-s} \in K$.

Define for $t \in [0, e^s]$

$$\beta(t) := \frac{t}{1 - e^{-s}rt},$$

$$g_t := \begin{pmatrix} (1 - e^{-s}rt)^{-1} & 0 \\ -e^{-s}r & 1 - e^{-s}rt \end{pmatrix}.$$

It is easy to see $d(e, g_t) < \epsilon_3$. And we have for every $t \in [0, e^s]$,

$$(5.10) \quad u_{-e^{-s}r}^+ u_t = u_{\beta(t)} g_t.$$

Step 2. For $t \in [0, e^s]$, if $xa_{-s}u_t g_t^{-1}, xa_{-s}u_t \in \Omega$, then for every $i \in \{1, \dots, k\}$

$$d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+ u_t, \Phi(xa_{-s})u_t) < 2\epsilon_2.$$

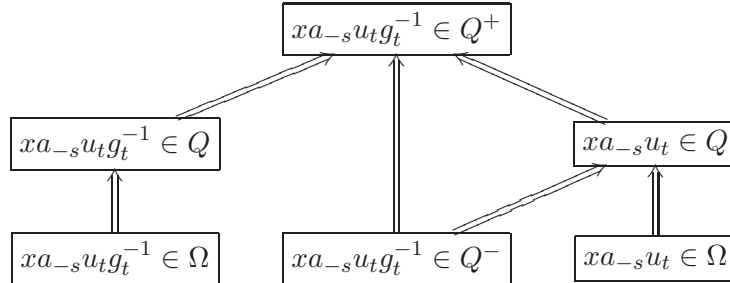
In fact, we can obtain this inequality by using (5.10)

$$\begin{aligned} & d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+ u_t, \Phi(xa_{-s})u_t) \\ &= d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{\beta(t)} g_t, \Phi(xa_{-s})u_t) \\ &= d(\varphi_{j(i)}(xa_{-s}u_{e^{-s}r}^+ u_{\beta(t)})g_t, \Phi(xa_{-s}u_t)) \quad (\text{by the } U\text{-equivariance}) \\ &\leq d(\varphi_{j(i)}(xa_{-s}u_t g_t^{-1})g_t, \varphi_{j(i)}(xa_{-s}u_t)) \\ &< 2\epsilon_2. \end{aligned}$$

Step 3. We claim the following inequality holds:

$$\frac{|\{t \in [0, e^s] : xa_{-s}u_t, xa_{-s}u_t g_t^{-1} \in \Omega\}|}{|\{t \in [0, e^s] : xa_{-s}u_t \in \Omega\}|} \geq 2 \cdot (1 - \epsilon_1)^5 \cdot (1 - \epsilon_2)^2 - 1.$$

Let Q^+, Q^- be the sets defined as (5.4). We may assume that $\epsilon_3 < \text{diam}(U)$. For every $t \in [0, e^s]$, since $d(e, g_t) < \epsilon_3$, we have the following relations:



Then

$$\begin{aligned}
& \frac{\int_0^{e^s} \chi_\Omega(xa_{-s}u_t g_t^{-1}) dt}{\int_0^{e^s} \chi_{Q^+}(xa_{-s}u_t g_t^{-1}) dt} \\
&= \frac{\int_0^{e^s} \chi_\Omega(xu_r^+ a_{-s} u_{\beta(t)}) dt}{\int_0^{e^s} \chi_{Q^+}(xu_r^+ a_{-s} u_{\beta(t)}) dt} \quad (\text{by (5.10)}) \\
&= \frac{\int_0^{\frac{e^s}{1-r}} \chi_\Omega(xu_r^+ a_{-s} u_l) \cdot (1 - e^{-s}rt)^2 dl}{\int_0^{\frac{e^s}{1-r}} \chi_{Q^+}(xu_r^+ a_{-s} u_l) \cdot (1 - e^{-s}rt)^2 dl} \quad (l = \beta(t)) \\
&\geq (1-r)^2 \cdot \frac{\int_0^{\frac{e^s}{1-r}} \chi_\Omega(xu_r^+ a_{-s} u_l) dl}{\int_0^{\frac{e^s}{1-r}} \chi_{Q^+}(xu_r^+ a_{-s} u_l) dl} \\
&\geq (1-\epsilon_2)^2 \cdot (1-\epsilon_1) \cdot \frac{m(\Omega)}{m(Q^+)} \quad (\text{since } xu_r^+ a_{-s} \in K \text{ and } s > T_1) \\
&\geq (1-\epsilon_2)^2 \cdot (1-\epsilon_1)^3.
\end{aligned}$$

And

$$\begin{aligned}
& \int_0^{e^s} \chi_\Omega(xa_{-s}u_t) dt \\
&\geq (1-\epsilon_1) \cdot \frac{m(\Omega)}{m(Q)} \cdot \int_0^{e^s} \chi_Q(xa_{-s}u_t) dt \\
&\geq (1-\epsilon_1)^2 \cdot \int_0^{e^s} \chi_{Q^-}(xa_{-s}u_t g_t^{-1}) dt \\
&\geq (1-\epsilon_1)^3 \cdot (1-\epsilon_2)^2 \cdot \frac{m(Q^-)}{m(Q^+)} \cdot \int_0^{e^s} \chi_{Q^+}(xa_{-s}u_t g_t^{-1}) dt \\
&\geq (1-\epsilon_1)^5 \cdot (1-\epsilon_2)^2 \cdot \int_0^{e^s} \chi_{Q^+}(xa_{-s}u_t g_t^{-1}) dt.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& |\{t \in [0, e^s] : xa_{-s}u_t, xa_{-s}u_t g_t^{-1} \in \Omega\}| \\
&\geq (2 \cdot (1-\epsilon_1)^5 \cdot (1-\epsilon_2)^2 - 1) \cdot |\{t \in [0, e^s] : xa_{-s}u_t g_t^{-1} \in Q^+\}| \\
&\geq (2 \cdot (1-\epsilon_1)^5 \cdot (1-\epsilon_2)^2 - 1) \cdot |\{t \in [0, e^s] : xa_{-s}u_t \in \Omega\}|.
\end{aligned}$$

The claim is justified.

Step 4. Let ρ be the constant given as (5.5). For each i , we claim that

$$\sup_{t \in [0, e^s]} (d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+u_t, \varphi_i(xa_{-s})u_t))^2 \leq \rho^2.$$

Set

$$\tilde{T} = \inf\{T \in [0, e^s] : (d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+u_T, \varphi_i(xa_{-s})u_T))^2 = \rho^2\}.$$

It follows from the choice of ϵ_2 and ϵ_3 in Step 1 that $\tilde{T} > \max\{T_0, T_1\}$. The polynomial divergence of horocycle flow implies that there exists a polynomial P of degree at most n_0 such that for every $t \in [0, \tilde{T}]$

$$P(t)/C_0 \leq (d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+u_t, \varphi_i(xa_{-s})u_t))^2 \leq C_0P(t).$$

Define

$$\Theta_{i,x}(t) := \min\{(d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+u_t, \Phi(xa_{-s})u_t))^2, \rho^2\}.$$

We have that for any $t \in [0, \tilde{T}]$, if $xa_{-s}u_t \in \Omega$, then

$$\Theta_{i,x}(t) = (d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+u_t, \varphi_i(xa_{-s})u_t))^2.$$

In fact, if there is another $j \neq i$ satisfying

$$\Theta_{i,x}(t) = (d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+u_t, \varphi_j(xa_{-s})u_t))^2,$$

then

$$d(\varphi_i(xa_{-s})u_t, \varphi_j(xa_{-s})u_t) \leq 2\rho.$$

However both $\varphi_i(xa_{-s})u_t$ and $\varphi_j(xa_{-s})u_t$ belong to the set $\Phi(xa_{-s}u_t)$. It follows from the property of Ω that this is a contradiction.

Since $\tilde{T} > T_0$, applying Lemma 4.2, we get

$$\begin{aligned} (5.11) \quad & \frac{\int_0^{\tilde{T}} \chi_\Omega(xa_{-s}u_t) \Theta_{i,x}(t) dt}{\int_0^{\tilde{T}} \chi_\Omega(xa_{-s}u_t) dt} \\ & \geq \frac{\int_0^{\tilde{T}} \chi_\Omega(xa_{-s}u_t) Q(t) dt}{C_0 \int_0^{\tilde{T}} \chi_\Omega(xa_{-s}u_t) dt} \\ & \geq \frac{C_1}{C_0} \cdot \sup_{t \in [0, \tilde{T}]} Q(t) \\ & \geq \frac{C_1}{C_0^2} \cdot \sup_{t \in [0, \tilde{T}]} (d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+u_t, \varphi_i(xa_{-s})u_t))^2. \end{aligned}$$

Meanwhile by the same argument as Steps 2 and 3, we have

$$\begin{aligned} (5.12) \quad & \frac{\int_0^{\tilde{T}} \chi_\Omega(xa_{-s}u_t) \Theta_{i,x}(t) dt}{\int_0^{\tilde{T}} \chi_\Omega(xa_{-s}u_t) dt} \\ & \leq 4\epsilon_2^2 + 2\rho^2 \cdot (1 - (1 - \epsilon_1)^5(1 - \epsilon_2)^2). \end{aligned}$$

If $\tilde{T} < e^s$, then

$$\sup_{t \in [0, \tilde{T}]} (d(\varphi_i(xa_{-s}u_{e^{-s}r}^+)u_{-e^{-s}r}^+u_t, \varphi_j(xa_{-s})u_t))^2 = \rho^2.$$

And (5.11) and (5.12) together imply that

$$\frac{C_1\rho^2}{C_0^2} \leq 4\epsilon_2^2 + 2\rho^2 \cdot (1 - (1 - \epsilon_1)^5(1 - \epsilon_2)^2),$$

contradicting (5.9). Therefore $\tilde{T} = e^s$ and the proof of Step 4 is completed.

Step 5. Completion of the proof of Proposition 5.8. Let $g_{s,i} \in G$ satisfying

$$\varphi_i(xu_r^+ a_{-s})u_{-e^{-s}r}^+ = \varphi_i(xa_{-s})g_{s,i}.$$

Step 4 in particular implies that $g_{s,i}$ is contained in an $O(1)$ neighborhood of the identity.

Write $g_{s,i} = \begin{pmatrix} x_s & y_s \\ z_s & w_s \end{pmatrix}$. Then

$$u_{-t}g_{s,i}u_t = \begin{pmatrix} x_s - tz_s & y_s + t(x_s - w_s) - t^2z_s \\ z_s & w_s + tz_s \end{pmatrix}.$$

Therefore it follows from Step 4 and the fact that $\det g_{s,i} = 1$ that

$$|z_s| = O(e^{-2s}), \quad |1 - x_s| = O(e^{-s}), \quad |1 - w_s| = O(e^{-s}), \quad |y_s| = O(1).$$

This implies

$$d(e, a_{-s}g_{s,i}a_s) = O(e^{-s}).$$

In consequence,

$$\begin{aligned} \varphi_i(xu_r^+ a_{-s})u_{-e^{-s}r}^+ &= \varphi_i(xa_{-s})g_{s,i} \\ &\in \Phi(xa_{-s})a_s(a_{-s}g_{s,i}a_s)a_{-s} \\ &\in \Phi(x) \cdot \{g \in G : d(g, e) = O(e^{-s})\} \cdot a_{-s}. \end{aligned}$$

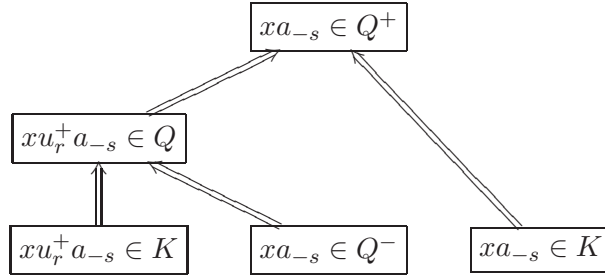
Noting that $\varphi_i(xu_r^+ a_{-s})u_{-e^{-s}r}^+ \in \Phi(xu_r^+)u_{-r}^+a_{-s}$, we conclude that

$$\Phi(xu_r^+)u_{-r}^+ \subset \Phi(x) \cdot \{g \in G : d(g, e) = O(e^{-s})\}.$$

This proves Proposition 5.8 with $r_0 = \epsilon_3$ (constructed in Step 1). \square

Proof of Theorem 5.1. Fix any $x \in \tilde{X}$ and $r \in (0, \epsilon_3)$ with $xu_r^+ \in \tilde{X}$. We show that there exists an increasing sequence $\{s_n\} \subset \mathbb{R}_{>0}$ such that $xa_{-s_n}, xu_r^+a_{-s_n} \in K$.

For any $s > 0$, noting that $d(e, u_{e^{-s}r}^+) < \epsilon_3$, we have the following relations:



By construction, every point in \tilde{X} is generic for the Hopf's ratio theorem for the geodesic flow with respect to the family of functions in \mathcal{F}_2 . For any

sufficiently large T , there exists a constant $c = c(T)$ such that

$$\begin{aligned}
\int_0^T \chi_K(xu_r^+ a_{-s}) ds &\geq c \cdot \frac{m(K)}{m(Q)} \cdot \int_0^T \chi_Q(xu_r^+ a_{-s}) ds \\
&\geq c \cdot (1 - \epsilon_1)^2 \cdot \int_0^T \chi_{Q^-}(xa_{-s}) ds \\
&\geq c^2 \cdot (1 - \epsilon_1)^2 \cdot \frac{m(Q^-)}{m(Q^+)} \cdot \int_0^T \chi_{Q^+}(xa_{-s}) ds \\
&\geq c^2 \cdot (1 - \epsilon_1)^4 \cdot \int_0^T \chi_{Q^+}(xa_{-s}) ds.
\end{aligned}$$

At the same time, we have

$$\begin{aligned}
\int_0^T \chi_K(xa_{-s}) ds &\geq c \cdot \frac{m(K)}{m(Q^+)} \cdot \int_0^T \chi_{Q^+}(xa_{-s}) ds \\
&\geq c \cdot (1 - \epsilon_1)^4 \cdot \int_0^T \chi_{Q^+}(xa_{-s}) ds.
\end{aligned}$$

It can be deduced from the above two inequalities that

$$\begin{aligned}
&|\{s \in [0, T] : xu_r^+ a_{-s}, xa_{-s} \in K\}| \\
&\geq (2c^2(1 - \epsilon_1)^3 - 1) \cdot |\{s \in [0, T] : xa_{-s} \in K\}|.
\end{aligned}$$

The right-hand side of the above inequality is greater than 0 because c is close to 1 when T is sufficiently large and

$$\int_0^\infty \chi_K(xa_{-s}) ds = \infty$$

by the property of \tilde{X} .

Therefore there exists an increasing sequence $\{s_n\} \subset \mathbb{R}_{>0}$ such that $xa_{-s_n}, xu_r^+ a_{-s_n} \in K$. Applying Proposition 5.8, we have

$$\Phi(xu_r^+)u_{-r}^+ \subset \Phi(x) \cdot \{g \in G : d(e, g) = O(e^{-s_n})\}.$$

As $s_n \rightarrow \infty$, this implies that

$$\Phi(xu_r^+)u_{-r}^+ = \Phi(x).$$

□

6. JOINING CLASSIFICATION

In this section, we prove the classification theorem of ergodic U -joinings (Theorem 1.4). The proof is divided into several steps. Let μ be any ergodic U -joining measure on $Z := X_1 \times X_2$. First we show that μ is invariant under the action of $\Delta(A)$ up to conjugation (Corollary 6.11): this consists of showing that μ is invariant under the action of a nontrivial connected subgroup of $\Delta(A)(\{e\} \times U)$ (Theorem 6.3) and that μ cannot be invariant under $\{e\} \times U$ (Lemma 6.10). Next we prove that there exist a conull set $\Omega \subset Z$ and a positive integer l so that $\#\pi_1^{-1}(x^1) \cap \Omega = l$ for m_{Γ_1} -a.e. $x^1 \in X_1$,

where $\pi_1 : Z \rightarrow X_1$ is the canonical projection (Theorem 6.13). This will yield an AU -equivariant set-valued map $\mathcal{Y} : X_1 \rightarrow X_2$. Applying Theorem 5.1 to \mathcal{Y} , we prove that there exists $q_0 \in G$ so that $\Gamma_2 q_0 \Gamma_1 = \cup_{j=1}^l \Gamma_2 q_0 \gamma_j$ with $\gamma_j \in \Gamma_1$ and

$$\mathcal{Y}(\Gamma_1 g) = \{\Gamma_2 q_0 \gamma_1 g, \dots, \Gamma_2 q_0 \gamma_l g\},$$

for m_{Γ_1} -a.e. $\Gamma_1 g$ (Proposition 6.15). This will eventually imply that μ is in fact a finite cover self-joining (Definition 1.3), completing the proof of Theorem 1.4.

6.1. $\Delta(A)$ -invariance of μ . Fix the followings:

- (1) a non-negative function $\psi \in C_c(X_1)$ with $m_{\Gamma_1}(\psi) > 0$ and set

$$\Psi = \psi \circ \pi_1 \in C(Z);$$

- (2) a compact subset $\Omega \subset X_1$ so that Theorems 3.3 and 3.5 hold uniformly for ψ for all $x \in \Omega$;
 (3) a constant

$$0 < r := \frac{1}{4}r(\frac{1}{2}; \Omega) < 1,$$

where $r(\frac{1}{2}; \Omega)$ is given as Theorem 3.3;

- (4) a compact subset $Q \subset \Omega \times X_2$ such that for any $x \in Q$, every $f \in C_c(Z)$ and $g \in C_c(X_1)$, the following holds:

$$(6.1) \quad \lim_{T \rightarrow \infty} \frac{\int_0^T f(x\Delta(u_t))dt}{\int_0^T g \circ \pi_1(x\Delta(u_t))dt} = \frac{\mu(f)}{\mu(g \circ \pi_1)}.$$

Fix a small $\epsilon > 0$ and choose $\eta > 0$ small enough so that $\mu(Q\{g : |g| < \eta\}) \leq (1 + \epsilon)\mu(Q)$. We put

$$Q_+ := Q\{g : |g| \leq \eta/4\} \text{ and } \mathcal{F} = \{\chi_Q, \chi_{Q_+}\}.$$

As every point Q satisfies Theorem 3.3 as well as (6.1), a simple computation yields

$$(6.2) \quad \lim_{T \rightarrow \infty} \frac{\int_{rT}^T f(x\Delta(u_t))dt}{\int_{rT}^T \Psi(x\Delta(u_t))dt} = \frac{\mu(f)}{\mu(\Psi)}$$

holds for every $f \in \mathcal{F}$ and for μ -a.e. $x \in Q$. Set Q_ϵ to be a compact subset in Q with $\mu(Q_\epsilon) > (1 - \epsilon)\mu(Q)$ so that (6.2) converges uniformly on Q_ϵ .

Denote by $N_{G \times G}(\Delta(U))$ the normalizer of $\Delta(U)$ in $G \times G$.

Theorem 6.3. *Let $h_k \in G \times G - N_{G \times G}(\Delta(U))$ be a sequence tending to e as $k \rightarrow \infty$. If $Q_\epsilon h_k \cap Q_\epsilon \neq \emptyset$ for every k , then μ is invariant under a nontrivial connected subgroup of $\Delta(A)(\{e\} \times U)$. Moreover, if $\{h_k\}$ contains a subsequence in $\{e\} \times G$, then μ is invariant under $\{e\} \times U$.*

Given Theorems 3.3 and 3.5 in our setting, the proof of Theorem 7.12 in [24] works here. For readers' convenience, we sketch the proof.

Lemma 6.4 (Lemma 7.7 in [24]). *If $h \in N_{G \times G}(\Delta(U))$ satisfies $Q_\epsilon h \cap Q_\epsilon \neq \emptyset$, then μ is h -invariant.*

Proof of Theorem 6.3. Letting $h_k = (h_k^1, h_k^2)$ and $h_k^i = \begin{pmatrix} a_k^i & b_k^i \\ c_k^i & d_k^i \end{pmatrix}$ for $i = 1, 2$, define for $t \neq -d_k^1/c_k^1$

$$\alpha_k(t) = \frac{b_k^1 + a_k^1 t}{d_k^1 + c_k^1 t}.$$

Let x_k be a point in Q_ϵ so that $y_k = x_k h_k \in Q_\epsilon$. We can write

$$y_k \Delta(u_t) = x_k h_k \Delta(u_t) = x_k \Delta(u_{\alpha_k(t)}) \varphi_k(t)$$

for some $\varphi_k(t) \in AU^+ \times G$. Associated to $\varphi_k(t)$'s, we obtain a quasi-regular map $\varphi : \mathbb{R} \rightarrow \Delta(A)(\{e\} \times U)$ satisfying

$$\varphi(t) := \lim_k \varphi_k(R_k t),$$

where $\{R_k\}$ is a sequence of positive numbers tending to ∞ as $k \rightarrow \infty$. We refer readers to Section 7.1 in [24] or Section 5 in [22] for details.

Fix some sufficiently small $\sigma > 0$. Since $h_k \rightarrow e$ as $k \rightarrow \infty$, we can find an increasing sequence $\{T_k\}$ such that for all large k , the derivative of α_k satisfies

$$(6.5) \quad 1 - \sigma \leq \alpha'_k(t) \leq 1 + \sigma \text{ for any } t \in [0, T_k].$$

We claim that there exist constants $c_1 > 1$ and $\tilde{T} = \tilde{T}(Q_\epsilon, \Psi) > 1$ so that for all large k and for every $f \in \mathcal{F}$,

$$(6.6) \quad c_1^{-1} \int_{rT}^T f(x \Delta(u_t)) dt \leq \int_{rT}^T f(x \Delta(u_{\alpha_k(t)})) dt \leq c_1 \int_{rT}^T f(x \Delta(u_t)) dt$$

holds for all $x \in Q_\epsilon$ and $T \in (\tilde{T}, T_k)$.

As $\alpha_k(0) \rightarrow 0$ as $k \rightarrow \infty$ and $\alpha'_k(t)$ is close to 1 for $t \in [0, T_k]$ ((6.5)), replacing t by $\alpha_k(t)$, there exists $T_0 > 1$ such that for all large k , any $T \in [T_0, T_k]$, any $f \in \mathcal{F} \cup \{\Psi\}$ and $x \in Q_\epsilon$,

$$\begin{aligned} & (1 + \sigma)^{-1} \int_{(1+2\sigma)rT}^{(1-2\sigma)T} f(x \Delta(u_t)) dt \\ & \leq \int_{rT}^T f(x \Delta(u_{\alpha_k(t)})) dt \\ & \leq (1 - \sigma)^{-1} \int_{(1-2\sigma)rT}^{(1+2\sigma)T} f(x \Delta(u_t)) dt. \end{aligned}$$

Applying Theorems 3.3 and 3.5 to the first and third equations in above inequalities for $f = \Psi$, we can verify that the claim is valid for Ψ . Note that the limit (6.2) converges uniformly on Ω_ϵ , we conclude that there are constants $c_1 > 1$ and $\tilde{T} = \tilde{T}(Q_\epsilon, \Psi) > 1$ so that for all large k and for every $f \in \mathcal{F}$, (6.6) holds for all $x \in Q_\epsilon$ and $T \in (\tilde{T}, T_k)$.

Set τ'_k to be the infimum of $\tau > 0$ such that

$$\sup_{t \in [0, \tau]} d(e, \varphi_k(t)) = \eta/4,$$

and put $\tau_k = \min\{\tau'_k, T_k\}$. Note that $\theta_k = \tau_k/R_k$ is bounded away from 0. Passing to a subsequence if necessary, we may assume θ_k 's converge to some $\theta \neq 0$.

Let $T' > 1$ be a constant satisfying for all $T > T'$ and for every $z \in Q_\epsilon$,

$$(6.7) \quad \frac{\int_{rT}^T \chi_Q(z\Delta(u_t))dt}{\int_{rT}^T \chi_{Q^+}(z\Delta(u_t))dt} > 1 - \epsilon.$$

Note that we have the following relations:

$$x_k\Delta(u_{\alpha_k(t)}) \in Q \Rightarrow y_k\Delta(u_t) \in Q_+ \Leftarrow y_k\Delta(u_t) \in Q.$$

The lower bound for the amount of time when $x_k\Delta(u_{\alpha_k(t)}) \in Q$ is given as follows:

$$(6.8) \quad \begin{aligned} & \int_{rT}^T \chi_Q(x_k\Delta(u_{\alpha_k(t)}))dt \\ & \geq c_1^{-1} \int_{rT}^T \chi_Q(x_k\Delta(u_t))dt \\ & \geq c_1^{-1}(1 - \epsilon) \int_{rT}^T \chi_{Q^+}(x_k\Delta(u_t))dt \\ & \geq c_1^{-1}(1 - \epsilon) \int_{rT}^T \chi_Q(y_k\Delta(u_{\alpha_k^{-1}(t)}))dt \\ & \geq c_1^{-2}(1 - \epsilon) \int_{rT}^T \chi_Q(y_k\Delta(u_t))dt \\ & \geq c_1^{-2}(1 - \epsilon)^2 \int_{rT}^T \chi_{Q^+}(y_k\Delta(u_t))dt. \end{aligned}$$

We can give a lower bound for $|\{t \in [rT, T] : y_k\Delta(u_t) \in Q\}|$ in terms of $|\{t \in [rT, T] : y_k\Delta(u_t) \in Q_+\}|$ using (6.7).

These relations together imply that for all large k and all $T \in [T', T_k]$

$$(6.9) \quad \{t \in [rT, T] : x_k\Delta(u_{\alpha_k(t)}), y_k\Delta(u_t) \in Q\} > 0.$$

Now for each k , let m_k be the largest integer so that $r^{m_k}\tau_k > T_0$. Then for any $l \geq 0$, we have $l \leq m_k$ holds for all large k . Applying (6.9) for $T_{k,l} = r^l\tau_k$, we obtain $t \in [r^{l+1}\tau_k, r^l\tau_k]$ and $z_{k,l} \in Q$ with $z_{k,l}\varphi_k(t) \in Q$. Passing to a subsequence we get $z_l \in Q$ and $s \in [r^{l+1}\theta, r^l\theta]$ so that $z_l\varphi(s) \in Q$. Therefore μ is $\varphi(s)$ -invariant by Lemma 6.4. If l is large enough, then $\varphi(s) \neq e$ gets arbitrarily close to e . The first claim of the theorem is proved noticing that the image of φ is contained in $\Delta(A)(\{e\} \times G)$.

As for the second claim, the construction of φ (see Section 7.1 for details) indicates that the image of φ is contained in $N_{G \times G}(\Delta(U)) \cap (\{e\} \times G)$ if

$\{h_k\} \subset \{e\} \times G$. Consequently, under this situation, the joining measure μ is invariant under $\{e\} \times U$. \square

The following lemma follows from the proof of Lemma 7.16 in [24]:

Lemma 6.10. *The ergodic joining measure μ is not invariant under $\{e\} \times U$.*

Now we draw the following corollary from Theorem 6.3 and Lemma 6.10:

Corollary 6.11. *The ergodic joining measure μ is invariant under a non-trivial connected subgroup A' of $\Delta(A)(\{e\} \times U)$ which is not contained in $\{e\} \times U$.*

Proof. Keep the same notations as in Theorem 6.3. In particular, Q is a compact subset with $\mu(Q) > 0$ and $Q_\epsilon \subset Q$ with $\mu(Q_\epsilon) \geq (1 - \epsilon)\mu(Q)$.

Let $\pi_i : Z \rightarrow X_i$ be the canonical projection for $i = 1, 2$. Since $(\pi_i)_*\mu = m_{\Gamma_i}$ and m_{Γ_i} does not support on proper Zariski subvarieties, we can choose sequences $\{x_k\}, \{y_k\} \subset Q_\epsilon$ so that $y_k = x_k h_k$ with $h_k \notin N_{G \times G}(\Delta(U))$ and h_k tends to e as $k \rightarrow \infty$.

Applying Theorem 6.3 to $\{h_k\}$, we get a map

$$\varphi : \mathbb{R} \rightarrow N_{G \times G}(\Delta(U)) \cap \mathcal{L} = \Delta(A) \cdot (\{e\} \times U)$$

so that μ is invariant under a non-trivial connected subgroup L in the image of φ . The corollary follows from Lemma 6.10. \square

By replacing μ by $(e, u) \cdot \mu$, we may assume that μ is $\Delta(AU)$ -invariant in the rest of the section.

6.2. Finiteness of fiber measures. Let $\mathcal{P}(X_2)$ be the set of probability measures. By the standard disintegration theorem, there exists an m_{Γ_1} -conull set $X'_1 \subset X_1$ and a measurable function $X'_1 \rightarrow \mathcal{P}(X_2)$ given by $x^1 \mapsto \mu_{x^1}^{\pi_1}$ such that for any Borel subsets $Y \subset Z$ and $C \subset X_1$,

$$(6.12) \quad \mu(Y \cap \pi_1^{-1}(C)) = \int_C \mu_{x^1}^{\pi_1}(Y) dm_{\Gamma_1}(x^1).$$

The measure $\mu_{x^1}^{\pi_1}$ is called the fiber measure over $\pi_1^{-1}(x^1)$.

Theorem 6.13. *There exist a positive integer l and an m_{Γ_1} -conull subset $X' \subset X_1$ so that $\text{supp}(\mu_{x^1}^{\pi_1})$ is a finite set with cardinality l for all $x^1 \in X'$. Furthermore,*

$$\mu_{x^1}^{\pi_1}(x^2) = 1/l$$

for any $x^1 \in X'$ and $x^2 \in \text{supp} \mu_{x^1}^{\pi_1}$.

Proof. This theorem can be regarded as a corollary of Theorem 6.3. It follows from the proof of Theorem 7.17 in [24]. \square

6.3. Reduction to the rigidity of measurable factors. By Theorem 6.13, there exists a conull set $\tilde{X} \subset X_1$ and a positive integer l so that $\mu_{x^1}^{\pi_1}$ is supported on l points for every $x^1 \in \tilde{X}$.

Define a set-valued map $\mathcal{Y} : \tilde{X} \rightarrow X_2$ given by

$$(6.14) \quad \mathcal{Y}(x^1) = \text{supp } \mu_{x^1}^{\pi_1}.$$

It follows from [33] that there are measurable maps

$$\varphi_1, \dots, \varphi_l : \tilde{X} \rightarrow X_2$$

so that $\mathcal{Y}(x^1) = \{\varphi_1(x^1), \dots, \varphi_l(x^1)\}$ for $x^1 \in \tilde{X}$. Furthermore, noting that μ is $\Delta(AU)$ -invariant, by possibly changing $\{\mu_{x^1}^{\pi_1}\}$ on a set of m_{Γ_1} -measure zero, we may assume that \mathcal{Y} is defined on X_1 and it is AU -equivariant.

Proposition 6.15. *Let $\mathcal{Y} : X_1 \rightarrow X_2$ be defined as (6.14). In particular, we have that \mathcal{Y} is AU -equivariant. Then there exists $q_0 \in G$ so that $[\Gamma_1 : \Gamma_1 \cap q_0^{-1}\Gamma_2q_0] = l$. Putting $\Gamma_2q_0\Gamma_1 = \bigcup_{j=1}^l \Gamma_2q_0\gamma_j$ with $\gamma_j \in \Gamma_1$, we have*

$$\mathcal{Y}(\Gamma_1g) = \{\Gamma_2q_0\gamma_1g, \dots, \Gamma_2q_0\gamma_lg\}$$

for m_{Γ_1} -a.e. Γ_1g .

Lemma 6.16. *There exists a set-valued map $\mathcal{Y}_0 : X_1 \rightarrow X_2$ so that \mathcal{Y}_0 is G -equivariant and it agrees with \mathcal{Y} on a conull set of X_1 .*

Proof. Applying Theorem 5.1 to \mathcal{Y} , we obtain a conull subset $\tilde{X} \subset X_1$ so that for every $x^1 \in \tilde{X}$ and every $u_r^+ \in U^+$ with $xu_r^+ \in \tilde{X}$,

$$\mathcal{Y}(xu_r^+) = \mathcal{Y}(x)u_r^+.$$

Using Fubini theorem, we know that for m_{Γ_1} -a.e. $x \in X_1$,

$$(6.17) \quad \int_{\mathbb{R}} \chi_{\tilde{X}^c}(xu_r^+) dr = 0.$$

Fix $x_0 \in \tilde{X}$ so that x_0 satisfies (6.17). Denote $U^+(x_0) = \{u_r^+ : x_0u_r^+ \in \tilde{X}\}$. Identifying U^+ with \mathbb{R} , $U^+(x_0)$ is a conull set in U^+ .

Define another set-valued map $\mathcal{Y}_0 : X_1 \rightarrow X_2$ by $\mathcal{Y}_0(x_0g) = \mathcal{Y}(x_0)g$. We need to verify that \mathcal{Y}_0 is well-defined. We first show that \mathcal{Y}_0 is well-defined on x_0U^+AU . It suffices to show that for any two points $x_0u_{r_1}^+a_su_t$ and $x_0u_{r_2}^+$, if $x_0u_{r_1}^+a_su_t = x_0u_{r_2}^+$, then

$$\mathcal{Y}(x_0)u_{r_1}^+a_su_t = \mathcal{Y}(x_0)u_{r_2}^+.$$

Since $U^+(x_0)$ is a conull set in U^+ , there exists $u_r^+ \in U^+$ satisfying

- $x_0u_{r+r_2}^+ \in \tilde{X}$;
- $u_{r_1}^+a_su_tu_r^+ = u_{r'}^+a_{s'}u_{t'}$ with $u_{r'}^+ \in U^+(x_0)$.

We have

$$\mathcal{Y}(x_0)u_{r_1}^+a_su_tu_r^+ = \mathcal{Y}(x_0u_{r'}^+a_{s'}u_{t'}) = \mathcal{Y}(x_0u_{r+r_2}^+) = \mathcal{Y}(x_0)u_{r+r_2}^+,$$

which implies \mathcal{Y}_0 is well-defined on x_0U^+AU .

Next we show that \mathcal{Y}_0 is well-defined on X_1 . Suppose $x_0g = x_0$. We prove

$$\mathcal{Y}(x_0)g = \mathcal{Y}(x_0).$$

Let $\{g_n\}$ be a sequence in U^+AU tending g as $n \rightarrow \infty$. For every $i \in \{1, \dots, l\}$, we have

$$\begin{aligned} d(\varphi_i(x_0), \mathcal{Y}(x_0)g) &= \min_{1 \leq j \leq l} d(\varphi_i(x_0), \varphi_j(x_0)g) \\ &\leq \min_{1 \leq j \leq l} (d(\varphi_i(x_0), \varphi_j(x_0)g_n) + d(\varphi_j(x_0)g_n, \varphi_j(x_0)g)) \\ &\leq d(\varphi_i(x_0), \mathcal{Y}(x_0)g_n) + d(g_n, g) \\ &= d(\varphi_i(x_0), \mathcal{Y}_0(x_0g_n)) + d(g_n, g). \end{aligned}$$

Observe that U^+AU contains a neighborhood V of e in G . There exists a sequence $\{h_n\}$ in V so that $x_0h_n = x_0g_n$ and h_n tends to e as $n \rightarrow \infty$. This implies that

$$\begin{aligned} d(\varphi_i(x_0), \mathcal{Y}(x_0)g) &\leq d(\varphi_i(x_0), \mathcal{Y}_0(x_0h_n)) + d(g_n, g) \\ &= d(\varphi_i(x_0), \mathcal{Y}(x_0)h_n) + d(g_n, g) \\ &\leq d(\varphi_i(x_0), \varphi_i(x_0)h_n) + d(g_n, g) \\ &\rightarrow 0 \end{aligned} \quad \text{as } n \rightarrow \infty.$$

Therefore \mathcal{Y}_0 is well-defined and \mathcal{Y}_0 agrees with \mathcal{Y} on $x_0U^+(x_0)AU$. \square

Proof of Proposition 6.15. By Lemma 6.16, we can show the proposition for \mathcal{Y}_0 . Let $x_0 \in X_1$ be the point given in the proof of Lemma 6.16. Write $x_0 = \Gamma_1g_0$ and

$$\mathcal{Y}_0(\Gamma_1g_0) = \{\Gamma_2h_1, \dots, \Gamma_2h_l\}.$$

The G -equivariance of \mathcal{Y}_0 implies $\mathcal{Y}_0(\Gamma_1g_0)$ is $g_0^{-1}\Gamma_1g_0$ -invariant. Putting $q_i = h_i g_0^{-1}$ for $i = 1, \dots, l$, we have for every i

$$(6.18) \quad \Gamma_2q_i\Gamma_1 \subset \{\Gamma_2q_1, \dots, \Gamma_2q_l\}.$$

This implies $\Gamma_1 \cap q_i^{-1}\Gamma_2q_i$ is a finite index subgroup of Γ_1 .

Fixing i , assume that $[\Gamma_1 : \Gamma_1 \cap q_i^{-1}\Gamma_2q_i] = l_i \leq l$. In view of (6.18), we have

$$\Gamma_2q_i\Gamma_1 = \{\Gamma_2q_{i_1}, \dots, \Gamma_2q_{l_i}\}.$$

Consider the set

$$X_i := \{(x^1, x^2) : x^1 = \Gamma_1g, x^2 \in \{\Gamma_2q_{i_1}g, \dots, \Gamma_2q_{l_i}g\}\}.$$

Observe the set

$$X = \{(x^1, x^2) : x^1 = \Gamma_1g, x^2 \in \mathcal{Y}_0(x^1) = \{\Gamma_2q_1g, \dots, \Gamma_2q_lg\}\},$$

is a conull set for the joining measure μ since \mathcal{Y}_0 agrees with \mathcal{Y} almost everywhere. Then $\mu_{x^1}^{\pi_1}(X_i) = l_i/l$ for m_{Γ_1} -a.e. $x^1 \in X_1$. As X_i is $\Delta(U)$ -invariant set with positive measure, we conclude $l_i = l$ and X_i agrees with X up to sets of measure zero. Therefore $q_i \in G$ is an element satisfying Proposition 6.15. \square

Proof of Theorem 1.4. Keep the notations in Proposition 6.15. In particular, let $q_0 \in G$ be an element satisfying Proposition 6.15 so that $\Gamma_2 q_0 \Gamma_1 = \cup_{j=1}^l \Gamma_2 q_0 \gamma_j$ with $\gamma_j \in \Gamma_1$.

For the ergodic U -joining measure μ , recall the disintegration of μ in terms of $\mu_{\Gamma_1 g}^{\pi_1}$ (6.12). It follows from Proposition 6.15 that $\mu_{\Gamma_1 g}^{\pi_1}$ is a uniformly distributed on $\{\Gamma_2 q_0 \gamma_1 g, \dots, \Gamma_2 q_0 \gamma_l g\}$ for m_{Γ_1} -a.e. $\Gamma_1 g$. This implies that μ is $\Delta(G)$ -invariant.

Letting $\Gamma_0 = \Gamma_1 \cap q_0^{-1} \Gamma_2 q_0$, the map

$$\psi : \Gamma_0 \backslash G \rightarrow \Gamma_1 \backslash G \times \Gamma_2 \backslash G$$

given by $\Gamma_0 g \mapsto (\Gamma_1 g, \Gamma_2 q_0 g)$ provides a homeomorphism between $\Gamma_0 \backslash G$ and its image. Then the pullback of μ through ψ provides a G -invariant measure on $\Gamma_0 \backslash G$. Therefore μ is a multiple of the pushforward of m_{Γ_0} through ψ .

Now we show Γ_0 is also a finite index subgroup of $q_0^{-1} \Gamma_2 q_0$. Choose a neighborhood B of e in G so that $B \cap q_0^{-1} \Gamma_2 q_0 = \{e\}$. Up to scalars, we have

$$\begin{aligned} m_{\Gamma_2}(\Gamma_2 q_0 B) &= \mu(\Gamma_1 \backslash G \times \Gamma_2 q_0 B) \\ &= m_{\Gamma_0}(\psi^{-1}(\Gamma_1 \backslash G \times \Gamma_2 q_0 B)) \\ &= \sum_{\alpha} m_{\Gamma}(\Gamma \gamma_{\alpha} B), \end{aligned}$$

where $\{\Gamma \gamma_{\alpha}\}_{\alpha}$ are the cosets of Γ in $q_0^{-1} \Gamma_2 q_0$. Since $m_{\Gamma_2}(\Gamma_2 q_0 B) < \infty$, this equality implies that Γ_0 is a finite index subgroup of $q_0^{-1} \Gamma_2 q_0$.

In conclusion, the ergodic U -joining measure μ is a finite cover self-joining (Definition 1.3). \square

Remark 6.19. We provide a proof here showing that the U -action on $\Gamma_0 \backslash G$ is ergodic with respect to m_{Γ_0} . To see this, note that Γ_1 is of divergent type by Rees ([31]). Hence Γ_0 , as a finite index subgroup of Γ_1 , is also of divergent type. Any non-elementary discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ has non-arithmetic length spectrum. Therefore the ergodicity of m_{Γ_0} with respect to U -action can be deduced from the works of Kaimanovich ([15]) and Roblin ([32]).

Now we deduce Corollary 1.7 as a corollary of Theorem 1.4.

Proof of Corollary 1.7. Denote by π the projection from $\Gamma_1 \backslash G \times \Gamma_2 \backslash G$ to $\Gamma_1 \backslash G$. Let μ be any $\Delta(AU)$ -invariant, ergodic, conservative, infinite Radon measure on the product space. Then the pushforward of μ through π , denoted by $(\pi)_* \mu$, is a $\Delta(AU)$ -invariant, ergodic measure on $\Gamma_1 \backslash G$. It follows from the main theorem in [1] that $(\pi)_* \mu = m_{\Gamma_1}$. Applying the disintegration theorem to μ , we have

$$\mu = \int_{x \in \Gamma_1 \backslash G} \mu_x dm_{\Gamma_1}(x),$$

where μ_x is a probability measure on $\{x\} \times \Gamma_2 \backslash G$ for m_{Γ_1} -a.e. x .

The discrepancy of μ is determined by whether μ is invariant under $\{e\} \times U$ or not. Suppose μ is not invariant under $\{e\} \times U$. Note that in the proof of Theorem 1.4, we require the ergodic U -joining is not invariant under $\{e\} \times U$ (Corollary 6.11). Now applying Theorem 1.4 to μ , we conclude that μ is of the form described in case (2).

If μ is invariant under $\{e\} \times U$, then μ_x is a $\{e\} \times U$ -invariant on $\{x\} \times \Gamma_2 \backslash G$ for m_{Γ_1} -a.e. x . By the unique ergodicity of U on $\Gamma_2 \backslash G$ ([11]), we have $\mu_x = m_{\Gamma_2}$ for m_{Γ_1} -a.e. x . Hence $\mu = m_{\Gamma_1} \times m_{\Gamma_2}$.

Next we show that $m_{\Gamma_1} \times m_{\Gamma_2}$ is $\Delta(AU)$ -ergodic. Suppose $m_{\Gamma_1} \times m_{\Gamma_2}$ is not $\Delta(AU)$ -ergodic. Let τ be any ergodic component in the ergodic decomposition of $m_{\Gamma_1} \times m_{\Gamma_2}$. Then τ is conservative under the action of $\Delta(AU)$ for $(\pi)_* \tau = m_{\Gamma_1}$. The above analysis implies τ should be of the form described in Corollary 1.7 (2).

Now set

$$\text{Comm}(\Gamma_1; \Gamma_2) = \{g \in G : [\Gamma_1 : \Gamma_1 \cap g^{-1}\Gamma_2g] < \infty\}.$$

Since Γ_1 and Γ_2 are countable, there exists a countable field k so that $\Gamma_i \subset \text{SL}_2(k)$ for $i = 1, 2$. For every $g \in \text{Comm}(\Gamma_1; \Gamma_2)$, we have that $g \in \text{SL}_2(k)$ by Chapter VII, Lemma 6.2 in [21]. (In fact, the proof of the lemma is valid as long as Γ_1 and Γ_2 are Zariski dense.) This implies that the set $\text{Comm}(\Gamma_1; \Gamma_2)$ is countable.

Note that $m_{\Gamma_1} \times m_{\Gamma_2}$ gives measure zero to the sets of the form

$$([e], [g])\Delta(G)(\{e\} \times AU),$$

where $g \in \text{Comm}(\Gamma_1; \Gamma_2)$. Then $m_{\Gamma_1} \times m_{\Gamma_2}$ is a zero measure by the countability of $\text{Comm}(\Gamma_1; \Gamma_2)$, which is a contradiction. Therefore, we have that the action of $\Delta(AU)$ is ergodic with respect to $m_{\Gamma_1} \times m_{\Gamma_2}$. \square

7. U -FACTOR CLASSIFICATION

Let Γ be a \mathbb{Z} or \mathbb{Z}^2 -cover. This section is devoted to proving Corollary 1.6. Given a U -equivariant factor map $p : (\Gamma \backslash G, m_\Gamma) \rightarrow (Y, \nu)$, consider the following map

$$\begin{aligned} \Gamma \backslash G &\rightarrow Y \times \Gamma \backslash G \\ [g] &\mapsto (p([g]), [g]). \end{aligned}$$

The pushforward of m_Γ through this map, denoted by μ , is an ergodic U -joining measure with respect to the pair of measures (ν, m_Γ) . And μ can be disintegrated into the following form:

$$(7.1) \quad \mu = \int_{y \in Y} \tau_y d\nu(y),$$

where τ_y is a probability measure supported on $\{y\} \times p^{-1}(y)$ for ν -a.e. y .

We first show that the measure τ_y is fully atomic for ν -a.e. y .

Proposition 7.2. *Under the assumption of Corollary 1.6, there exist a conull set Ω in $\Gamma \backslash G$ and a positive integer l_0 so that $\#p^{-1}(y) \cap \Omega = l_0$ for ν -a.e. y . Furthermore, the measure τ_y is uniform distributed on $\{y\} \times (p^{-1}(y) \cap \Omega)$ for ν -a.e. y .*

Proof. The proof is parallel to the proof of Theorem 6.13. The key lies in obtaining window property I (Theorem 3.3) for Y using the factor map p . We claim that τ_y is fully atomic for ν -a.e. y , or equivalently, the set

$$B' = \{y \in Y : \tau_y \text{ is not fully atomic}\}$$

is a null set.

Suppose the claim fails. Then $\nu(B') > 0$. For every $y \in B'$, decompose τ_y into the following form:

$$\tau_y = (\tau_y)^a + (\tau_y)^c,$$

where $(\tau_y)^a$ and $(\tau_y)^c$ are respectively the purely atomic part and the continuous part of τ_y . Let

$$B = \{(y, [g]) \in Y \times \Gamma \backslash G : y \in B' \text{ and } [g] \in \text{supp}(\tau_y)^c\}.$$

We will construct two compact subsets Q and Q_ϵ in B as Section 6.1. To be precise, fix a nonnegative function $\psi \in C_c(Y)$ with $\nu(\psi) > 0$. Then $\psi \circ p \in L^1(\Gamma \backslash G, m_\Gamma)$. Let π_1 be the canonical projection from $Y \times \Gamma \backslash G$ to Y . Set

$$\Psi = \psi \circ \pi_1 \in C(Y \times \Gamma \backslash G).$$

Choose a compact subset D in $p^{-1}(B')$ so that $p|_D$ is continuous and the window property I (Theorem 3.3) holds for $\psi \circ p$ uniformly for all $[g] \in D$. Let

$$0 < r := \frac{1}{4}r(1/2; D) < 1$$

be the constant given as Theorem 3.3. As a result, there exists $T_0 > 1$ so that we have for every $T > T_0$ and for every $(y, [g]) \in p(D) \times \Gamma \backslash G \cap B$

$$(7.3) \quad \int_0^{rT} \Psi((y, [g])\Delta(u_t))dt \leq \frac{1}{2} \int_0^T \Psi((y, [g])\Delta(u_t))dt.$$

Set Q to be a compact subset in $p(D) \times \Gamma \backslash G \cap B$ so that the following holds for every $(y, [g]) \in Q$ and for every $f \in C_c(Y \times \Gamma \backslash G)$:

$$\lim_{T \rightarrow \infty} \frac{\int_0^T f((y, [g])\Delta(u_t))dt}{\int_0^T \Psi((y, [g])\Delta(u_t))dt} = \frac{\mu(f)}{\mu(\Psi)}.$$

Fix a small $\epsilon > 0$ and choose $\eta > 0$ small enough so that $\mu(Q\{(e, g) : g \in G, |g| \leq \eta\}) < (1 + \epsilon)\mu(Q)$. Set

$$Q_+ = Q\{(e, g) : g \in G, |g| \leq \eta/4\}.$$

In view of (7.3), we have for μ -a.e. $(y, [g]) \in Q$

$$(7.4) \quad \lim_{T \rightarrow \infty} \frac{\int_{rT}^T \chi_Q((y, [g])\Delta(u_t))dt}{\int_{rT}^T \chi_{Q_+}((y, [g])\Delta(u_t))dt} = \frac{\mu(Q)}{\mu(Q_+)}.$$

Let $Q_\epsilon \subset Q$ be a compact subset so that $\mu(Q_\epsilon) > (1 - \epsilon)\mu(Q)$ and (7.4) converges uniformly on Q_ϵ .

If the claim fails, there exists a sequence $\{(y, [g_k])\} \subset Q_\epsilon$ converging to some point $(y, [g]) \in Q_\epsilon$. This is because Q_ϵ is a subset of B and applying Fubini's theorem to $\mu(Q_\epsilon)$, we have

$$\mu(Q_\epsilon) = \int_{y \in B'} (\tau_y)^c(Q_\epsilon) d\nu(y) > 0.$$

Write $(y, [g_k]) = (y, [g])(e, h_k)$ where $h_k \neq e$ and $h_k \rightarrow e$ as $k \rightarrow \infty$. Then $Q_\epsilon h_k \cap Q_\epsilon \neq \emptyset$. Applying the argument of Theorem 6.3 to Q_ϵ (Theorem 7.12 in [24]), we deduce that there exists a sequence $\{(e, u_k)\} \subset \{e\} \times U$ converging to (e, e) so that $Q_\epsilon(e, u_k) \cap Q_\epsilon \neq \emptyset$. This implies μ is invariant under $\{e\} \times U$ (cf. Lemma 7.7 in [24]). However, it follows from the proof as Lemma 7.16 in [24] that μ cannot be invariant under $\{e\} \times U$. Therefore, the measure τ_y is fully atomic for ν -a.e. y .

Now set

$$\Omega' = \{(y, [g]) \in Y \times \Gamma \backslash G : \tau_y([g]) = \max_{[g'] \in p^{-1}(y)} \tau_y([g'])\}.$$

This is a $\Delta(U)$ -invariant set of positive μ -measure. The ergodicity of μ yields that Ω' is a conull set. Moreover, there exists a positive integer l_0 so that τ_y is uniform distributed on l_0 -points. Let π_2 be the canonical projection from $Y \times \Gamma \backslash G$ to $\Gamma \backslash G$. Then $\Omega := \pi_2(\Omega')$ is a conull set satisfying Proposition 7.2. \square

Denote the Haar measure on G by \tilde{m} . Let $\text{Comm}_G(\Gamma)$ be the commensurator subgroup of Γ in G , that is, $g \in \text{Comm}_G(\Gamma)$ if and only if Γ and $g^{-1}\Gamma g$ are commensurable with each other.

Lemma 7.5. *For $i = 1, 2$, let $h_i \in \text{Comm}_G(\Gamma)$, $u_i \in U$, and φ_i be the map*

$$\Gamma \cap h_i^{-1}\Gamma h_i \backslash G \rightarrow \Gamma \backslash G \times \Gamma \backslash G$$

given by $[g] \mapsto ([g], [h_i g u_i])$. Set $\mu_i = (\varphi_i)_ m_{\Gamma \cap h_i^{-1}\Gamma h_i}$. If μ_1 is not proportional to μ_2 , then*

$$\Gamma h_1 g u_1 \neq \Gamma h_2 g u_2$$

for \tilde{m} -a.e. g .

Proof. Set

$$W = \{g \in G : \Gamma h_1 g u_1 = \Gamma h_2 g u_2\}.$$

We show that W is a null set in G . Suppose W is of positive measure.

Let $\Gamma_i = \Gamma \cap h_i^{-1}\Gamma h_i$ and $\rho_i : G \rightarrow \Gamma_i \backslash G$ be the natural projection. Consider the following diagram:

$$\begin{array}{ccc}
 & G & \\
 \rho_1 \swarrow & & \searrow \rho_2 \\
 \Gamma_1 \backslash G & & \Gamma_2 \backslash G \\
 \varphi_1 \searrow & & \swarrow \varphi_2 \\
 & \Gamma \backslash G \times \Gamma \backslash G &
 \end{array}$$

We have $\varphi_1 \circ \rho_1|_W = \varphi_2 \circ \rho_2|_W$. Observe that W is a conull set because $\rho_1(W)$ is U -invariant and m_{Γ_1} is U -ergodic (Remark 6.19). When restricting μ_1 and μ_2 to $\varphi_1 \circ \rho_1(W)$, any μ_1 -measure zero set A is also μ_2 -measure zero. Hence we can consider the Radon-Nikodym derivative $d\mu_2/d\mu_1$. Note that $d\mu_2/d\mu_1$ is $\Delta(U)$ -invariant. Therefore, $\mu_1 = c\mu_2$ for some $c > 0$, which is a contradiction. \square

Proof of Corollary 1.6. Follow the notations in Proposition 7.2. Recall the measures τ_y 's given as (7.1). Denote by σ_y the pushforward measure $(\pi_2)_*\tau_y$, where π_2 is the canonical projection from $Y \times \Gamma \backslash G$ to $\Gamma \backslash G$. Consider the following measure on $\Gamma \backslash G \times \Gamma \backslash G$:

$$\bar{\mu} = \int_{y \in Y} \sigma_y \otimes \sigma_y d\nu(y),$$

where $\sigma_y \otimes \sigma_y$'s are the product measures on $\Gamma \backslash G \times \Gamma \backslash G$. The measure $\bar{\mu}$ is a U -joining measure with respect to the pair (m_Γ, m_Γ) . Let Ω be the conull subset given by Proposition 7.2. The set

$$\Omega \times_p \Omega := \{(x_1, x_2) \in \Omega \times \Omega : p(x_1) = p(x_2)\}$$

is a $\bar{\mu}$ -conull set. We claim that there exist finitely many $h_1, \dots, h_k \in \text{Comm}_G(\Gamma)$ and $u_1, \dots, u_k \in U$ so that up to sets of measure zero

$$\Omega \times_p \Omega = \cup_{1 \leq i \leq k} [(e, h_i)]\Delta(G)(e, u_i).$$

Let μ_Δ be the U -ergodic measure on $\Gamma \backslash G \times \Gamma \backslash G$ attained by pushing forward the Haar measure m_Γ on $\Gamma \backslash G$ through the diagonal embedding:

$$\begin{aligned}
 \Gamma \backslash G &\rightarrow \Gamma \backslash G \times \Gamma \backslash G \\
 [g] &\mapsto ([g], [g]).
 \end{aligned}$$

If $\bar{\mu}$ equals a multiple of μ_Δ , the claim is obvious. Now suppose $\bar{\mu}$ is not a multiple of μ_Δ . Consider the $\Delta(U)$ -ergodic decomposition of $\bar{\mu}$:

$$\bar{\mu} = \int_{z \in Z} \mu_z d\sigma(z),$$

where (Z, σ) is a probability space. For σ -a.e. z , the measure μ_z is an ergodic U -joining measure so that $\Omega \times_p \Omega$ is a μ_z -conull set. Choose any ergodic component μ_1 that is not a multiple of μ_Δ . Applying the joining

classification theorem (Theorem 1.4) to μ_1 , there exist $h_1 \in \text{Comm}_G(\Gamma)$ and $u_1 \in U$ so that up to a scalar, μ_1 is the pushforward of $m_{\Gamma \cap h_1^{-1}\Gamma h_1}$ through the map

$$\begin{aligned} \Gamma_1 \cap h_1^{-1}\Gamma h_1 \backslash G &\rightarrow \Gamma \backslash G \times \Gamma \backslash G \\ [g] &\mapsto ([g], [h_1 g u_1]). \end{aligned}$$

Since $\Omega \times_p \Omega$ is a μ_1 -conull set, we have $p(\Gamma g) = p(\Gamma h_1 g u_1)$ and $\tau_{p(\Gamma g)}(\Gamma h_1 g u_1) = 1/l_0$ for \tilde{m} -a.e. g . Let $i_1 = [\Gamma : \Gamma \cap h_1^{-1}\Gamma h_1]$. By Lemma 7.5, for ν -a.e. y ,

$$\sigma_y \otimes \sigma_y([(e, e)]\Delta(G) \cup [(e, h_1)]\Delta(G)(e, u_1)) = (i_1 + 1)/l_0.$$

If $i_1 + 1 < l_0$, choose another ergodic component μ_2 of $\bar{\mu}$ so that μ_2 is a U -ergodic joining measure and $\Omega \times_p \Omega$ is a μ_2 -conull set. The claim can be verified by repeating the above process finitely many times.

The sets $\{h_1, \dots, h_k\} \subset \text{Comm}_G(\Gamma)$ and $\{u_1, \dots, u_k\} \subset U$ yield a set $\{c_1 = e, \dots, c_n\}$ and a set $\{u_{p_1} = e, \dots, u_{p_n}\}$ satisfying:

$$(7.6) \quad \begin{aligned} &\text{for every } c_i \text{ and every } \gamma, c_i \gamma \in \Gamma c_j \text{ for some } j; \\ &p^{-1}(p(\Gamma g)) \cap \Omega = \{\Gamma c_1 g u_{p_1}, \dots, \Gamma c_n g u_{p_n}\} \text{ for } \tilde{m}\text{-a.e. } g. \end{aligned}$$

We show that $p_1 = p_2 = \dots = p_n = 0$.

Fix any $s \neq 0$. For \tilde{m} -a.e. g , we have

$$\begin{aligned} p^{-1}(\Gamma g a_s) \cap \Omega &= \{\Gamma c_1 g a_s u_{p_1}, \dots, \Gamma c_n g a_s u_{p_n}\} \\ &= \{\Gamma c_1 g u_{p_1} a_s u_{b_1}, \dots, \Gamma c_n g u_{p_n} a_s u_{b_n}\}, \end{aligned}$$

where $b_i = p_i(1 - e^{-s})$ for $1 \leq i \leq n$.

Set $B = \{b_1, \dots, b_n\}$. For m_{Γ} -a.e. $x, y \in \Gamma \backslash G$, if $p(x) = p(y)$, then $p(x a_s) = p(y a_s u_{b(y, x)})$ for some $b(y, x) \in B$ and $p(y a_s) = p(x a_s u_{b(x, y)})$ for some $b(x, y) \in B$. Since p is U -equivariant, we get

$$(7.7) \quad b(x, y) = -b(y, x).$$

This implies for \tilde{m} -a.e. $x, y, z \in \Gamma \backslash G$, if $p(x) = p(y) = p(z)$, then

$$(7.8) \quad b(x, z) = b(y, z) - b(y, x).$$

Suppose there exists $p_i \neq 0$. Then $b_i = p_i(1 - e^{-s}) \neq 0$. Denote $\bar{b} = \max\{b_1, \dots, b_n\}$ and $\tilde{b} = \min\{b_1, \dots, b_n\}$. Let $x, y, z \in \Gamma \backslash G$ be such that

$$p(x) = p(y) = p(z) \text{ and } b(y, z) = \bar{b}, b(y, x) = \tilde{b}.$$

Then (7.7) and (7.8) imply

$$b(x, z) = b(y, z) - b(y, x) = \bar{b} - \tilde{b} = 2\bar{b} > \bar{b},$$

which contradicts the maximality of \bar{b} . Hence $p_1 = \dots = p_n = 0$.

Now for \tilde{m} -a.e. g and for every $1 \leq i \leq n$, we have

$$\begin{aligned} p^{-1}(p(\Gamma g)) \cap \Omega &= \{\Gamma c_1 g, \dots, \Gamma c_n g\}, \\ p^{-1}(p(\Gamma c_i^{-1} g)) \cap \Omega &= \{\Gamma c_1 c_i^{-1} g, \dots, \Gamma c_n c_i^{-1} g\}, \\ p^{-1}(p(\Gamma c_i g)) \cap \Omega &= \{\Gamma c_1 c_i g, \dots, \Gamma c_n c_i g\}. \end{aligned}$$

So for every $i, j \in \{1, \dots, n\}$, we have $c_i^{-1} \in \Gamma c_j$ and $c_i c_j \in \Gamma c_l$ for some $1 \leq l \leq n$. Let Γ_0 be the group generated by Γ and $\{c_1, \dots, c_n\}$. We deduce from the above relation between Γ and $\{c_1, \dots, c_n\}$ together with (7.6) that Γ is a finite index subgroup of Γ_0 . The proof is completed. \square

APPENDIX: \mathbb{Z}^d -COVER GROUP ORBITS IN COMPACT HYPERBOLIC SURFACES

Let Γ_1 be a \mathbb{Z} or \mathbb{Z}^2 -cover and let Γ_2 be a cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$. We show the following theorem:

Theorem 7.9. *Any Γ_1 -orbit on $\Gamma_2 \backslash \mathrm{PSL}_2(\mathbb{R})$ is either finite or dense.*

When Γ_1 is a non-elementary finitely generated discrete subgroup, such an orbit classification theorem is shown by Benoist-Quint [4] using the classification of stationary measures. Later, Benoist and Oh provided an elementary and topological proof [5], inspired by the work of McMullen-Mohammadi-Oh [23]. Our proof of Theorem 7.9 is modeled on Benoist-Oh's proof. In particular, Theorem 7.9 can be deduced from the following Theorem 7.10 (see [5] for the deduction). Let

$$\begin{aligned} G &:= \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}), \\ H &:= \{(h, h) : h \in \mathrm{PSL}_2(\mathbb{R})\}, \\ \Gamma &:= \Gamma_1 \times \Gamma_2. \end{aligned}$$

Theorem 7.10. *For any $x \in \Gamma \backslash G$, the orbit xH is either closed or dense.*

7.1. Dynamics of unipotent flows. A key input in the proof of Theorem 7.10 is the window property of the horocycle flow on $\Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$ (Theorem 3.3). Set

- $N := \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\};$
- $D := \{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R}\}, A = \{(a_t, a_t)\};$
- $U_1 = \{(u_t, e)\}, U_2 = \{(e, u_t)\}, U = \{(u_t, u_t)\}.$

For simplicity, we write \tilde{u}_t for (u_t, u_t) and \tilde{a}_t for (a_t, a_t) .

Definition 7.11. Let $K > 1$. A subset $T \subset \mathbb{R}$ is called K -thick if T meets $[-Kt, -t] \cup [t, Kt]$ for all $t > 0$.

Denote the Haar measure on $\Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$ by m_{Γ_1} . The following proposition can be easily deduced from Theorem 3.3.

Proposition 7.12. *For any compact subset Q_1 in $\Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$ with $m_{\Gamma_1}(Q_1) > 0$, there exist a compact subset $Q_2 \subset Q_1$ of positive measure and constants $K, T_0 > 1$ such that for $Q_1(T_0) = \bigcup_{-T_0 \leq t \leq T_0} Q_1 u_t$, the set*

$$\{t \in \mathbb{R} : xu_t \in Q_1(T_0)\}$$

is K -thick for every $x \in Q_2$.

7.2. Proof of Theorem 7.10. Let $X = \Gamma \backslash G$. Our proof is modeled on [5] using the U -minimal sets relative to a fixed compact subset of X . In the construction of minimal sets, we need to find a compact subset $\Omega \subset X$ such that the U -orbit of every element of Ω returns to Ω for K -thick amount of time for some $K > 1$. When Γ_1 is finitely generated, there is a natural compact subset in X to use, which is the non-wandering set of the geodesic flow. When it comes to our setting, such a non-wandering set is the whole X and hence non-compact. In view of Proposition 7.12, instead of finding one such compact subset, we construct two compact subsets $\Omega_2 \subset \Omega_1$ in X such that the U -orbit of every element of Ω_2 returns to Ω_1 for K -thick amount of time for some $K > 1$. This difference results in some modification in the statement. But with Proposition 7.12 available, the proof is essentially a verbatim repetition of Benoist-Oh's proof. We will list the steps of the proof and point out the necessary modification.

Set Q'_1 to be a compact subset in $\Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$ of positive measure such that for every point $x_1 \in Q'_1$, the orbit $x_1 N$ is dense in $\Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$. It is shown in [19] that for m_{Γ_1} -a.e. x_1 , the orbit $x_1 N$ is dense in $\Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$. Hence such a compact set Q'_1 exists.

Let Q_2 be a compact subset in Q'_1 such that for every $x_1 \in Q_2$, the set $\{t \in \mathbb{R}_{\geq 0} : x_1 a_t \in Q'_1\}$ is unbounded and the set $\{t \in \mathbb{R} : x_1 u_t \in \cup_{|t| \leq T_0} Q'_1 u_t\}$ is K -thick for some constants $K, T_0 > 1$. The existence of such a compact set Q_2 follows from Proposition 7.12 and the fact that the D -action on $\Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$ is conservative ([31]).

Let $Q_1 = \cup_{|t| \leq T_0} Q'_1 u_t$. Set

$$\Omega_1 := Q_1 \times \Gamma_2 \backslash \mathrm{PSL}_2(\mathbb{R}) \text{ and } \Omega_2 := Q_2 \times \Gamma_2 \backslash \mathrm{PSL}_2(\mathbb{R}).$$

Note that for each $x \in \Omega_2$, the set

$$T(x, \Omega_1) := \{t \in \mathbb{R} : x \tilde{u}_t \in \Omega_1\}$$

is K -thick and the set $\{t \in \mathbb{R}_{\geq 0} : x \tilde{a}_t \in \Omega_1\}$ is unbounded.

Let $x = (x_1, x_2) \in X$ and consider the orbit xH . Let Y be an H -minimal subset of the closure \overline{xH} with respect to Ω_1 , i.e., Y is a closed H -invariant subset of \overline{xH} such that $Y \cap \Omega_1 \neq \emptyset$ and yH is dense in Y for every $y \in Y \cap \Omega_1$. Let Z be a U -minimal subset of \overline{xH} with respect to Ω_1 . Such minimal sets Y and Z exist as Ω_1 is compact and xH intersects Ω_1 non-trivially.

In the following, we assume that the orbit xH is not closed and show that xH is dense in X .

Lemma 7.13. *The set Z intersects Ω_2 non-trivially.*

Proof. Let $z = (z_1, z_2) \in Z \cap \Omega_1$ and $w_1 \in Q_2$. It follows from the construction of Q_1 that the orbit $z_1 N$ is dense in $\Gamma_1 \backslash \mathrm{PSL}_2(\mathbb{R})$. As a result, there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that $z_1 u_{t_n}$ converges to w_1 . Since $\Gamma_2 \backslash \mathrm{PSL}_2(\mathbb{R})$ is compact, the sequence $\{z_2 u_{t_n}\}$ has a limit point $w_2 \in \Gamma_2 \backslash \mathrm{PSL}_2(\mathbb{R})$. Consequently, the point $(w_1, w_2) \in z\overline{U} \cap \Omega_2 = Z \cap \Omega_2$. \square

Theorem 7.10 follows from the similar argument as in [5]. In particular, we apply the proofs of Lemmas 3.3, 3.4 and Propositions 3.5, 3.6 in [5] to a point $z \in Z \cap \Omega_2$.

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